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A VORTEX MODEL FOR THE STUDY OF THE FLOW AT THE ROTOR BLADE OF A HELICOPTER

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A VORTEX MODLE FOR TREATING THE FLOW ON ROTOR BLADES OF A HELICOPTER

W. H. Isay

ABSTRACT. On the base of unsteady vortex lifting-line theory an approximate method to calculate the loading distribution on rotor blades in forward flight is presented. The theory takes account of the vortex wake geometry for non-uniform (example trapezoidal) flow through the rotor-disc as well as the effect of rolling up and contraction of free tip- and root-vortices is considered. Calculating the blade-circulation distribution requires careful attention to the case where the blades pass through the rolled-up tip- and root-vortex of the foregoing foil.

The appendix of this paper is concerned with the preparation of formulas to predict the compressible acoustic pressure field of the rotor.

1. INTRODUCTION AND SUMMARY /283*

Various approximation methods are given in the literature [1], [2], [3] for calculating the flow on rotor blades. They can be used for various (periodic or aperiodic) flight conditions, depending on the factors ignored. The available results are not completely satisfactory for any of these methods, and it therefore seems desirable to improve the method by considering effects which have not been considered up to the present.

The linearized extended lifting line theory has been developed in the most consistent way as far as the theoretical aspects are concerned. This is true

*Numbers in the margin indicate pagination in the original foreign text.

even though the simplifications which it contains (ignoring the deformation of the free vortex surfaces due to the nonuniform rotor flux and the flapping motion of the blades, contraction and rolling-up process of the free transverse vortices, the oblique and reverse flow at the blades for large propeller moduli) means that there is a considerable abstraction of physical reality [1], [2], [3].

In the present paper, a modified vortex model is given on the basis of the lifting line theory. This makes it possible to include some effects which were not included in previous analyses. Among these, we have the nonuniform flow through the rotor plane. In particular, we will now investigate the influence of the rolled-up and contracted tip and hub vortices of the leading blade have on the lift distribution of a blade, in the case where the considered target blade passes through them.

The flow boundary condition at the rotor blade leads to integral equations with new types of singularities, and this paper will basically deal with the theory of solution of these equations.

2. CONCEPT OF THE VORTEX MODEL

Let us assume that the rotor has N blades $(n = 0, 1, \ldots, N-1)$ and that there is a completely periodic flight state at the velocity w_0 in the direction of the negative z axis. The blade $n = 0$ is assumed to be the target blade at which the flow boundary conditions are to be satisfied. In this aircraft-fixed coordinate system, let $x = 0$ be the rotor plane(1). Therefore w_0 is the incident flow velocity in the z direction. In addition, u_0 is an incident flow in the x direction ($u_0^2 \ll w_0^2$) (Figure 1).

(1)We will either use Cartesian coordinates x, y, z or cylindrical coordinates x, y = r cos φ, z = r sin φ. In the same way, we will use s instead of r and ψ instead of φ as the integration variable for vortex lines. ω is the angular velocity and $\varphi_n = \varphi_0 + 2\pi n/N$ characterizes the instantaneous angular position of the n-th blade. R_i and R_0 are the inner and outer radii of the blade. Finally

$$\mathfrak{v} = u\,\mathfrak{e}_x + v\,\mathfrak{e}_y + w\,\mathfrak{e}_z = u\,\mathfrak{e}_x + V\,\mathfrak{e}_\varphi + W\,\mathfrak{e}_r$$

refers to the absolute velocity referred to the aircraft-fixed coordinate system. Otherwise, we will use the same notation of [1].

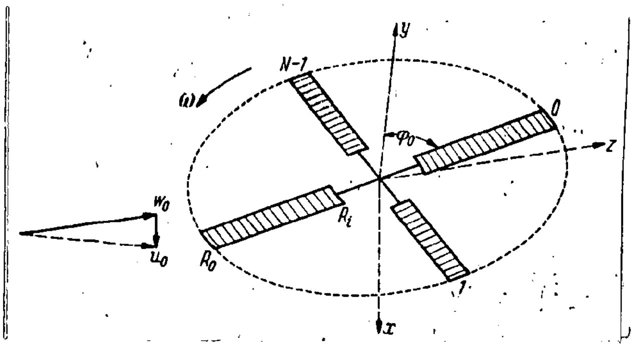

Fig. 1. Coordinate system of the rotor (N = 4).

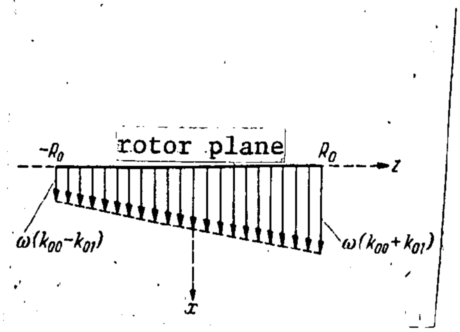

Fig. 2. Trapezoidal flow.

The free transverse and longitudinal vortices are located within an oblique cylinder behind the rotor which extends to infinity, just like in the linearized theory [1], [2], [3]. /284

Assumptions regarding the geometry of the free vortex surfaces will be made for the individual rotor blades. Common assumptions will be made for the wake regions as dictated by the physics of the problem.

Without considering the contraction and rolling-up process, a trapezoidal flow through the rotor plane in the x direction, such as

$$k_0(s,\psi) = k_{00} + k_{01}\frac{s}{R_0}\sin\psi, \quad k_{00} = \mathrm{const}; \quad k_{01} = \mathrm{const}; \tag{1}$$

to the free vortex surfaces having the shape (Figure 2);

$$\boldsymbol{r}_f = \psi\, k_0(s, \varphi_n + \psi)\, \boldsymbol{e}_x + s\cos(\varphi_n + \psi)\, \boldsymbol{e}_y + [s\sin(\varphi_n + \psi) + k_*\psi]\, \boldsymbol{e}_z \tag{2}$$

where

$$k_* = \mathrm{const}; \quad (0 \leqq \psi \leqq \infty; \quad R_i \leqq s \leqq R_0).$$

$$k_* \approx s\frac{w_0 + w_Q + w_L}{\omega s + V_Q + V_L} \approx \frac{w_0}{\omega}.$$

For a constant flux, we may assume k_0 = const. If there is no vortex motion through the rotor plane in certain regions, then we must set k_0 = 0 there.

The vortex axis vector of the free transverse vortices is given by $d\boldsymbol{s}_Q = \frac{\partial \boldsymbol{r}_f}{\partial\psi}d\psi$.

For the free longitudinal vortices it is $d\boldsymbol{s}_L = \frac{d\boldsymbol{r}_f}{\partial s}ds$.

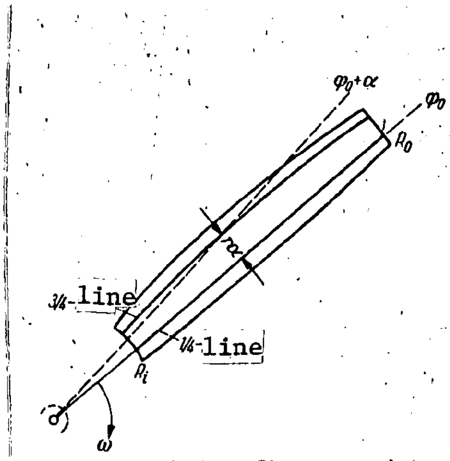

Fig. 3. Notations for the blade.

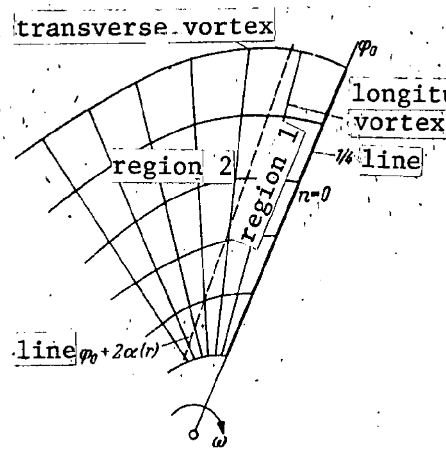

Fig. 4. Vortex system at the target blade. Region 1: no flow; region 2: trapezoidal flow.

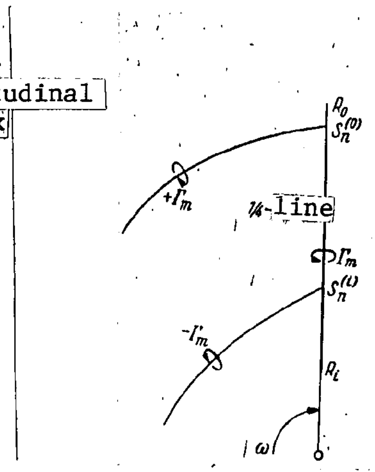

Fig. 5. Blade n ≠ 0. Rolled-up transverse vortex.

We will calculate the velocity field induced by the vortex system of the rotor along the 3/4 profile cord line of the target blade. It is required to satisfy the flow boundary conditions according to the extended lifting line theory (Figure 3).

The blades have the cord 2 r α(r), and their 3/4 line is given by x = 0; $y = r\cos(\varphi_n + \alpha)$, $z = r\sin(\varphi_n + \alpha)$, $\varphi_n = \varphi_0 + (2\pi n/N)$ ($n = 0$ target blade).

a)

We will replace the blades by bound rod vortices having the circulation $\Gamma(s, \varphi_n)$, $n = 0, 1, \ldots, N-1$, arranged in the rotor plane x = 0 at the 1/4 blade cord line. The induced velocity field along the 3/4 line of the target blade is then given by

$$v_\Gamma = \frac{1}{4\pi}\sum_{n=0}^{N-1}\int_{R_i}^{R_0} \Gamma(s, \varphi_n)\frac{r\sin\left(\alpha - \frac{2\pi n}{N}\right)\mathbf{e}_x\, ds}{\sqrt{r^2 + s^2 - 2rs\cos\left(\alpha - \frac{2\pi n}{N}\right)}^{3}}. \tag{3}$$

b)

We will select the following model for the free transverse vortex:

For the blade n = 0 (target blade) it is assumed that the transverse vortices remain behind the blade up to about 1/4 of the blade cord (that is up to $\psi \approx 2\alpha$)

without flowing through the blade plane x = 0 (linearized wing theory). After this, they leave the rotor plane (Figure 4) according to the trapezoidal flow distribution (1).

For the other blades (n ≠ 0), it is assumed that the transverse vortices have already been rolled up into a tip vortex (intensity $+\Gamma_m(\varphi_n+\psi)$, at $s=s_n^{(0)}<R_0$), and that there is a hub vortex (intensity $-\Gamma_m(\varphi_n+\psi)$, located at $s=s_n^{(i)}>R_i$). Γ_m is the maximum value of the bound circulation which corresponds to the instantaneous angular position $\varphi_n+\psi$ of the blade.(2) The radii $s_n^{(0)}$ and $s_n^{(i)}$ can be adjusted without difficulty to a wake contraction obtained from experiments. (Figure 5). /285

For the blade $n=N-1$, which preceeds the target blade being considered, we will assume that the tip and hub vortices remain approximately 1/4 blade chord behind the target blade in the rotor plane (Figure 6) and only then do they become pushed away corresponding to the trapezoidal flow (1). This vortex model also corresponds to the experience that the tip and hub vortices of the preceding blade $n=N-1$ greatly influences the flow conditions at the blade n = 0 being investigated. Often the latter passes through these rolled-up free vortices. One topic of this paper is to investigate this penetration effect and its influence on the lift distribution of a rotor blade. The vortex geometry described above is found to be especially suited for this purpose.

For the other blades n = 1 to $n=N-2$ we initially assume a flux according to equation (1) for the tip and hub vortices. The influence of these vorteces on the lift distribution at the blade n = 0 is smaller and penetration effects will hardly occur.

Under these conditions, one finds that the velocity field induced by the free transverse vortices at the 3/4 line of the target blade $x=0,\ y=r\cos(\varphi_0+\alpha)$, $z=r\sin(\varphi_0+\alpha)$ is given by:

(2) Instead of the maximum value, another circulation value can be substituted for Γ_m, which is better suited to the true flow conditions (for example, one taken from measurements). Instead of a trapezoidal distribution, any other flow distribution can be used.

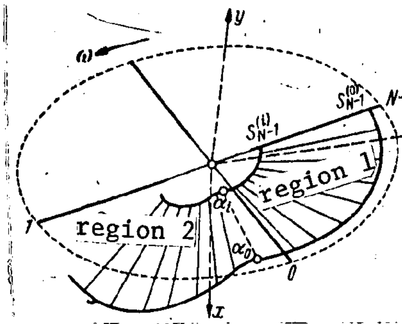

Fig. 6. Tip and hub vortices of the blade n = N - 1. Region 1: no flow, region 2: trapezoidal flow. Same for longitudinal vortex.

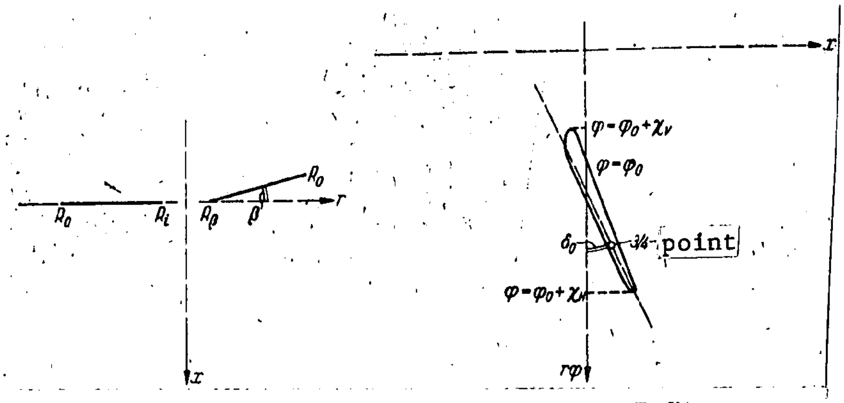

Fig. 7. a) Definition of the flapping angle, assumed, b) Blade cross section r = const.

$$\boldsymbol{v}_Q = \sum_{n=0}^{N-1} \boldsymbol{v}_Q^{(n)} .$$

$$\begin{aligned}
\boldsymbol{v}_Q^{(N-1)} = {} & \frac{1}{4\pi}\int_0^{\psi_0^{(0)}+\alpha_0} \Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\Bigg[\left(r\cos(\varphi_0+\alpha)-s_{N-1}^{(0)}\cos\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\right)^2+ \\
& +\left(r\sin(\varphi_0+\alpha)-s_{N-1}^{(0)}\sin\left(\varphi_0-\frac{2\pi}{N}+\psi\right)-k_*\psi\right)^2\Bigg]^{-3/2}\Bigg[(s_{N-1}^{(0)})^2-r s_{N-1}^{(0)}\cos\left(\alpha-\psi+\frac{2\pi}{N}\right)- \\
& -k_* r\cos(\varphi_0+\alpha)+k_* s_{N-1}^{(0)}\cos\left(\varphi_0-\frac{2\pi}{N}+\psi\right)+k_* s_{N-1}^{(0)}\psi\sin\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\Bigg]d\psi\, \boldsymbol{e}_x - \\
& -\frac{1}{4\pi}\int_0^{\psi_0^{(i)}+\alpha_i}\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi\right)[\text{Integrand with } s_{N-1}^{(i)}]\, d\psi\, \boldsymbol{e}_x + \\
& +\frac{1}{4\pi}\int_{\psi_0^{(0)}+\alpha_0}^{\infty}\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\Bigg[(\varkappa_{N-1}^{(0)})^2+\left(r\cos(\varphi_0+\alpha)-s_{N-1}^{(0)}\cos\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\right)^2+ \\
& +\left(r\sin(\varphi_0+\alpha)-s_{N-1}^{(0)}\sin\left(\varphi_0-\frac{2\pi}{N}+\psi\right)-k_*\psi\right)^2\Bigg]^{-3/2}\times \\
& \times\Bigg\{\Bigg[(s_{N-1}^{(0)})^2-r s_{N-1}^{(0)}\cos\left(\alpha-\psi+\frac{2\pi}{N}\right)-k_* r\cos(\varphi_0+\alpha)+k_* s_{N-1}^{(0)}\cos\left(\varphi_0-\frac{2\pi}{N}+\psi\right)+ \\
& +k_* s_{N-1}^{(0)}\psi\sin\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\Bigg]\boldsymbol{e}_x+\Bigg[\left(s_{N-1}^{(0)}\sin\left(\varphi_0-\frac{2\pi}{N}+\psi\right)-r\sin(\varphi_0+\alpha)+k_*\psi\right)\times \\
& \times\frac{\partial}{\partial\psi}\varkappa_{N-1}^{(0)}-\left(k_*+s_{N-1}^{(0)}\cos\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\right)\varkappa_{N-1}^{(0)}\Bigg]\boldsymbol{e}_y+\Bigg[\Bigg(r\cos(\varphi_0+\alpha)- \\
& -s_{N-1}^{(0)}\cos\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\Bigg)\frac{\partial}{\partial\psi}\varkappa_{N-1}^{(0)}-s_{N-1}^{(0)}\sin\left(\varphi_0-\frac{2\pi}{N}+\psi\right)\varkappa_{N-1}^{(0)}\Bigg]\boldsymbol{e}_z\Bigg\}d\psi - \\
& -\frac{1}{4\pi}\int_{\psi_0^{(i)}+\alpha_i}^{\infty}\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi\right)[\text{Integrand with } s_{N-1}^{(i)}]\, d\psi\ ;
\end{aligned} \qquad (4)$$

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with $\varkappa_{N-1}^{(0)} = (\psi - \psi_0^{(0)} - \alpha_0)\, k_0\left(s_{N-1}^{(0)}; \varphi_0 - \frac{2\pi}{N} + \psi\right)$; similarly $\varkappa_{N-1}^{(i)}$.

$$\sum_{n=1}^{N-2} v_Q^{(n)} = \frac{1}{4\pi} \sum_{n=1}^{N-2} \int_{\psi=0}^{\infty} \Gamma_m(\varphi_n + \psi)\, [(\varkappa_n^{(0)})^2 + ((r\cos(\varphi_0+\alpha) - s_n^{(0)}\cos(\varphi_n+\psi))^2 +$$
$$+ (r\sin(\varphi_0+\alpha) - s_n^{(0)}\sin(\varphi_n+\psi) - k_* \psi)^2]^{-3/2} \Bigg\{\Bigg[(s_n^{(0)})^2 - r\, s_n^{(0)} \cos\left(\alpha - \psi - \frac{2\pi n}{N}\right) -$$
$$- k_* r\cos(\varphi_0+\alpha) + k_* s_n^{(0)}\cos(\varphi_n+\psi) + k_* s_n^{(0)} \psi \sin(\varphi_n+\psi)\Bigg] e_x + \tag{5}$$
$$+ \left[\left(s_n^{(0)}\sin(\varphi_n+\psi) - r\sin(\varphi_0+\alpha) + k_*\psi\right)\frac{\partial}{\partial\psi}\varkappa_n^{(0)} - \left(k_* + s_n^{(0)}\cos(\varphi_n+\psi)\right)\varkappa_n^{(0)}\right] e_y +$$
$$+ \left[\left(r\cos(\varphi_0+\alpha) - s_n^{(0)}\cos(\varphi_n+\psi)\right)\frac{\partial}{\partial\psi}\varkappa_n^{(0)} - s_n^{(0)}\sin(\varphi_n+\psi)\,\varkappa_n^{(0)}\right] e_z\Bigg\} d\psi -$$
$$- \frac{1}{4\pi}\sum_{n=1}^{N-2}\int_{\psi=0}^{\infty} \Gamma_m(\varphi_n+\psi)\ [\text{Integrand with } s_n^{(i)}]\, d\psi\,;$$

with $\varkappa_n^{(0)} = \psi\, k_0(s_n^{(0)}, \varphi_n + \psi)$; $\quad \varkappa_n^{(i)} = \psi\, k_0(s_n^{(i)}, \varphi_n + \psi)$.

$$v_Q^{(0)} = \frac{1}{4\pi}\int_{R_i}^{R_0}\int_0^{2\alpha} \frac{\partial\Gamma(s, \varphi_0+\psi)}{\partial s}\, [(r\cos(\varphi_0+\alpha) - s\cos(\varphi_0+\psi))^2 +$$
$$+ (r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_*\psi)^2]^{-3/2}\, [r\, s\cos(\alpha-\psi) - s^2 +$$
$$+ k_* r\cos(\varphi_0+\alpha) - k_* s\cos(\varphi_0+\psi) - k_* s\,\psi\sin(\varphi_0+\psi)]\, d\psi\, ds\, e_x +$$
$$+ \frac{1}{4\pi}\int_{R_i}^{R_0}\int_{2\alpha}^{\infty} \frac{\partial\Gamma(s,\varphi_0+\psi)}{\partial s}\, [(\varkappa_0)^2 + (r\cos(\varphi_0+\alpha) - s\cos(\varphi_0+\psi))^2 + \tag{6}$$
$$+ (r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_*\psi)^2]^{-3/2} \times$$
$$\times \{[r\, s\cos(\alpha-\psi) - s^2 + k_* r\cos(\varphi_0+\alpha) - k_* s\cos(\varphi_0+\psi) - k_* s\,\psi\sin(\varphi_0+\psi)]\, e_x +$$
$$+ \left[\left(r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_*\psi\right)\frac{\partial\varkappa_0}{\partial\psi} + \left(k_* + s\cos(\varphi_0+\psi)\right)\varkappa_0\right] e_y +$$
$$+ \left[\left(s\cos(\varphi_0+\psi) - r\cos(\varphi_0+\alpha)\right)\frac{\partial\varkappa_0}{\partial\psi} + s\sin(\varphi_0+\psi)\,\varkappa_0\right] e_z\Big\}\, d\psi\, ds$$

with $\varkappa_0 = (\psi - 2\alpha)\, k_0(s, \varphi_0 + \psi)$.

In formula (4), $\psi_0^{(0)}$ and $\psi_0^{(i)}$ indicate the angles at which the tip and hub vortices of the blade $n = N - 1$ intersect the 3/4 line of the target blade n = 0. The corresponding radii are called r_0 and r_i. According to our definition we then have (Figure 6; Figure 8):

$$\left.\begin{aligned}
r_0\cos(\varphi_0+\alpha_0) &= s_{N-1}^{(0)}\cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0^{(0)}\right); \\
r_0\sin(\varphi_0+\alpha_0) &= s_{N-1}^{(0)}\sin\left(\varphi_0 - \frac{2\pi}{N} + \psi_0^{(0)}\right) + k_*\psi_0^{(0)}; \qquad \alpha_0 = \alpha(r_0)\,. \\
r_i\cos(\varphi_0+\alpha_i) &= s_{N-1}^{(i)}\cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0^{(i)}\right); \\
r_i\sin(\varphi_0+\alpha_i) &= s_{N-1}^{(i)}\sin\left(\varphi_0 - \frac{2\pi}{N} + \psi_0^{(i)}\right) + k_*\psi_0^{(i)}; \qquad \alpha_i = \alpha(r_i)\,.
\end{aligned}\right\} \tag{7}$$

c)

For the free longitudinal vortices we will assume the following model: for the blade n = 0, the vortices are arranged in the rotor plane up to 1/4 of the blade chord behind the blade (that is, up to $\psi = 2\alpha$) (linearized wing theory). After this they leave the plane x = 0 (Figure 4) according to the trapezoidal flux distribution (Equation 1).

The longitudinal vortices of the blades n = 1 to $n = N - 2$, which do not have a large influence on the flow boundary conditions, are extended after their creation outside of the rotor plane in the x direction according to the trapezoidal flux law.

The following two models will be discussed for treating the longitudinal vortices for the blade $n = N - 1$ preceding the target blade.

Case I: Vortex arrangement just as for the blades n = 1 to $n = N - 2$.

Case II: Just like the model used for the transverse vortices, the longitudinal vortices remain behind the target wing in the rotor plane x = 0 up to about 1/4 blade chord. They are blown away according to the flux given by equation (1) in the x direction (Figure 6). Using this concept, the velocity field induced by the free longitudinal vortices at the 3/4 line of the target wind x = 0, y = $r \cos(\varphi_0 + \alpha)$, $z = r \sin(\varphi_0 + \alpha)$ is given by

$$
\begin{aligned}
v_L &= \sum_{n=0}^{N-1} v_L^{(n)} \\
v_L^{(0)} &= \frac{1}{4\pi} \int_{R_i}^{R_a} \int_0^{2\alpha} \frac{\partial \Gamma(s, \varphi_0 + \psi)}{\partial \varphi_0} \left[\left(r \cos(\varphi_0 + \alpha) - s \cos(\varphi_0 + \psi) \right)^2 + \right. \\
&\quad \left. + \left(r \sin(\varphi_0 + \alpha) - s \sin(\varphi_0 + \psi) - k_* \psi \right)^2 \right]^{-3/2} \left[r \sin(\alpha - \psi) - k_* \psi \cos(\varphi_0 + \psi) \right] d\psi \, ds \, e_x + \\
&\quad + \frac{1}{4\pi} \int_{R_i}^{R_a} \int_{2\alpha}^{\infty} \frac{\partial \Gamma(s, \varphi_0 + \psi)}{\partial \varphi_0} \left[(\varkappa_0)^2 + \left(r \cos(\varphi_0 + \alpha) - s \cos(\varphi_0 + \psi) \right)^2 + \right. \\
&\quad \left. + \left(r \sin(\varphi_0 + \alpha) - s \sin(\varphi_0 + \psi) - k_* \psi \right)^2 \right]^{-3/2} \Big\{ \left[r \sin(\alpha - \psi) - k_* \psi \cos(\varphi_0 + \psi) \right] e_x + \\
&\quad + \left[\left(s \sin(\varphi_0 + \psi) - r \sin(\varphi_0 + \alpha) + k_* \psi \right) \frac{\partial \varkappa_0}{\partial s} - \sin(\varphi_0 + \psi) \varkappa_0 \right] e_y + \\
&\quad + \left[\left(r \cos(\varphi_0 + \alpha) - s \cos(\varphi_0 + \psi) \right) \frac{\partial \varkappa_0}{\partial s} + \cos(\varphi_0 + \psi) \varkappa_0 \right] e_z \Big\} \, d\psi \, ds \, ;
\end{aligned}
\tag{8}
$$

with $\varkappa_0 = (\psi - 2\alpha)\, k_0(s, \varphi_0 + \psi)$.

$$\sum_{n=1}^{N-2} \boldsymbol{v}_L^{(n)} = \frac{1}{4\pi} \sum_{n=1}^{N-2} \int_{R_i}^{R_a} \int_0^{\infty} \frac{\partial \Gamma(s, \varphi_n + \psi)}{\partial \varphi_0} [(\varkappa_n)^2 + (r\cos(\varphi_0 + \alpha) - s\cos(\varphi_n + \psi))^2 + \\ + (r\sin(\varphi_0 + \alpha) - s\sin(\varphi_n + \psi) - k_* \psi)^2]^{-3/2} \Big\{ \Big[r\sin\Big(\alpha - \psi - \frac{2\pi n}{N}\Big) - k_* \psi \cos(\varphi_n + \psi) \Big] \boldsymbol{e}_x + \\ + \Big[(s\sin(\varphi_n + \psi) - r\sin(\varphi_0 + \alpha) + k_* \psi) \frac{\partial \varkappa_n}{\partial s} - \sin(\varphi_n + \psi)\, \varkappa_n \Big] \boldsymbol{e}_y + \\ + \Big[(r\cos(\varphi_0 + \alpha) - s\cos(\varphi_n + \psi)) \frac{\partial \varkappa_n}{\partial s} + \cos(\varphi_n + \psi)\, \varkappa_n \Big] \boldsymbol{e}_z \Big\} \, d\psi\, ds \; ; \qquad (9)$$

with $\varkappa_n = \psi\, k_0(s, \varphi_n + \psi)$.

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$$\boldsymbol{v}_L^{(N-1)} = \frac{1}{4\pi} \int_{R_i}^{R_a} \int_0^{\alpha + \Psi_{N-1}} \frac{\partial \Gamma\Big(s, \varphi_0 - \frac{2\pi}{N} + \psi\Big)}{\partial \varphi_0} \Big[\Big(r\cos(\varphi_0 + \alpha) - s\cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big)\Big)^2 + \\ + \Big(r\sin(\varphi_0 + \alpha) - s\sin\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big) - k_* \psi\Big)^2 \Big]^{-3/2} \times \\ \times \Big[r\sin\Big(\alpha + \frac{2\pi}{N} - \psi\Big) - k_* \psi \cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big) \Big] d\psi\, ds\, \boldsymbol{e}_x + \\ + \frac{1}{4\pi} \int_{R_i}^{R_a} \int_{\alpha + \Psi_{N-1}}^{\infty} \frac{\partial \Gamma\Big(s, \varphi_0 - \frac{2\pi}{N} + \psi\Big)}{\partial \varphi_0} \Big[(\varkappa_{N-1})^2 + \Big(r\cos(\varphi_0 + \alpha) - s\cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big)\Big)^2 + \\ + \Big(r\sin(\varphi_0 + \alpha) - s\sin\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big) - k_* \psi\Big)^2 \Big]^{-3/2} \times \\ \times \Big\{ \Big[r\sin\Big(\alpha + \frac{2\pi}{N} - \psi\Big) - k_* \psi \cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big) \Big] \boldsymbol{e}_x + \\ + \Big[\Big(s\sin\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big) - r\sin(\varphi_0 + \alpha) + k_* \psi\Big) \frac{\partial \varkappa_{N-1}}{\partial s} - \sin\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big) \varkappa_{N-1} \Big] \boldsymbol{e}_y + \\ + \Big[\Big(r\cos(\varphi_0 + \alpha) - s\cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big)\Big) \frac{\partial \varkappa_{N-1}}{\partial s} + \cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi\Big) \varkappa_{N-1} \Big] \boldsymbol{e}_z \Big\} \, d\psi\, ds \; ; \qquad (10)$$

with $\varkappa_{N-1} = (\psi - \alpha - \Psi_{N-1})\, k_0\Big(s, \varphi_0 - \frac{2\pi}{N} + \psi\Big)$. Case II.

In Case I, $\boldsymbol{v}_L^{(N-1)}$ can be taken directly from Equation (9).

The angle Ψ_{N-1} in Equation (10) is defined by the singularity of the integrand of the first integral. The latter becomes singular when $\psi = \Psi_{N-1}$ and $s = s_{N-1}^*$. According to definition, we then have

$$
\left.\begin{aligned}
r\cos(\varphi_0+\alpha) &= s^*_{N-1}\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right);\\
r\sin(\varphi_0+\alpha) &= s^*_{N-1}\sin\left(\varphi_0-\frac{2\pi}{N}+\Psi^{N-1}\right)+k_*\,\Psi_{N-1}\,.
\end{aligned}\right| \qquad (11)
$$

This specifies the velocities induced by the vortex system of the rotor at the target blade.

Finally, according to convention, we will call β the flap angle of the blades and R_β is the distance of the flap joint from the rotor axis. $\omega(r-R_\beta)\left.\frac{d\beta}{d\varphi_0}\right|$ is the flapping velocity. Also δ_0 is the inclination angle of the profile skeleton line at the 3/4 point with respect to the $(r\varphi)$-axis (Figure 7).

The flow boundary condition is (Figure 3):

$$
\left.\begin{aligned}
&[\omega r+w_0\cos(\varphi_0+\alpha)]\tan\delta_0(r,\varphi_0)-u_0+(r-R_\beta)\,\omega\frac{d\beta}{d\varphi_0}\\
&\qquad= u_\Gamma+u_L+u_Q-(V_\Gamma+V_L+V_Q)\tan\delta_0(r,\varphi_0)\,;\\
&(R_i\leqq r\leqq R_0;\ 0\leqq\varphi_0'\leqq 2\pi)\ \big(V=-v\sin(\varphi_0+\alpha)+w\cos(\varphi_0+\alpha)\big)\,.
\end{aligned}\right| \qquad (12)
$$

After substituting the induced velocities, Equation (12) results in an integral equation for calculating the blade circulation.

The two flux constants k_{00} and k_{01} must be determined from an approximate calculation according to one of the known simple theories before Equation (12) can be solved. It can also be estimated. These values of k_{00} and k_{01} must be tested after solving the integral equation and after determining the induced velocities. If necessary, iterations must be performed.

3. INVESTIGATION AND TRANSFORMATION OF SINGULAR KERNEL PARTS AND BOUNDARY CONDITION INTEGRAL EQUATION /289

The representations (3) to (6) and (8) to (10) of the induced velocities already contain expressions which occur as kernels of the integral equation according to the boundary conditions (12). Therefore, we will not write them down again.

First of all, we will give a summary of the structure of these kernel components and will then develop a suitable solution theory. We must pay special attention to the influence on the circulation distribution of the fact that the considered blade penetrates the rolled-up free transverse vortex of the preceding blade.

First of all, the bound vortices according to Equation (3) as well as the sums

$$\sum_{n=1}^{N-2} v_Q^{(n)}, \quad \sum_{n=1}^{N-2} v_L^{(n)}$$

according to Equations (5) and (9) only result in continuous kernel components in the integral equation. Therefore, they must not be discussed.

For the velocities $v_Q^{(0)}, v_Q^{(N-1)}, v_L^{(0)}, v_L^{(N-1)}$, the integrands of the second integrals are always continuous. This is a result of formulas (4), (6), (8) and (10) and is a direct consequence of the physical vortex model used.

This means that the first integral components of the last mentioned velocities must be investigated in detail. They only contain an x component. These will be given the notation $(u_Q^{(0)})$, etc.

a)

We will consider the target blade n = 0.

The denominator of the part $(u_Q^{(0)} + u_L^{(0)})$ vanishes for prescribed target point coordinates r and $\varphi_0 + \alpha(r)$ when $s = s_0^*$ and $\psi = \Psi_0$. The quantities s_0^* and Ψ_0 are determined by the following equations:

$$\begin{aligned} r \cos(\varphi_0 + \alpha) &= s_0^* \cos(\varphi_0 + \Psi_0) \\ r \sin(\varphi_0 + \alpha) &= s_0^* \sin(\varphi_0 + \Psi_0) + k_* \Psi_0 \end{aligned} \qquad (13)$$

We will develop the numerator N and the denominator D of the integrand of the part $(u_Q^{(0)} + u_L^{(0)})$ in the vicinity of the point (13). We will use the substitution

$$\psi = \Psi_0 + \chi, \quad s = s_0^* + \sigma$$

after some elementary transformations, if third order terms are considered in the numerator and second order terms in χ and σ are considered in the denominator, we find:

$$D = [\sigma^2 + 2\sigma\chi k_* \sin(\varphi_0 + \Psi_0) + \chi^2 (s_0^{*2} + k_*^2 + 2 k_* s_0^* \cos(\varphi_0 + \Psi_0) - \\ - \chi^3 s_0^* k_* \sin(\varphi_0 + \Psi_0) + \sigma\chi^2 (s_0^* + 2 k_* \cos(\varphi_0 + \Psi))]^{3/2} ; \tag{14}$$

In a similar way, using the relationship (13):

$$N = -\frac{\partial\Gamma(s,\varphi_0+\Psi_0+\chi)}{\partial s}\left[\sigma(s_0^* + k_*\cos(\varphi_0+\Psi_0)) + \sigma^2 + \frac{1}{2}\chi^2 s_0^* (s_0^* + k_*\cos(\varphi_0+\Psi_0))\right] - \\ -\frac{\partial\Gamma(s,\varphi_0+\Psi_0+\chi)}{\partial\varphi_0}[\chi(s_0^* + k_*\cos(\varphi_0+\Psi_0)) - \chi^2 k_* \sin(\varphi_0+\Psi_0)] . \tag{15}$$

We will now introduce the following abbreviations:

$$A_0 = s_0^{*2} + k_*^2 + 2 k_* s_0^* \cos(\varphi_0 + \Psi_0) ; \quad B_0 = k_* \sin(\varphi_0 + \Psi_0) ; \\ \Delta_0^2 = (s_0^* + k_* \cos(\varphi_0 + \Psi_0))^2 = A_0 - B_0^2 . \tag{16}$$

In order to obtain an overview of the values of s_0^* and Ψ_0 as a solution of the Equations (13), we should consider the fact that in the limiting case $k_* \to 0$ we have

$$s_0^* = r ; \quad \Psi_0 = \alpha . \tag{17}$$

If we now consider small propeller moduli k_*/R_0, just as in the lifting line theory of the rotor and if we assume that there is no reverse flow, then we have $r + k^* \cos(\varphi_0 + \alpha) > 0$ for all blade target points. If we also consider the fact that $\alpha^2 \ll 1$, then it is appropriate to solve Equation (13) in the linearized form. We obtain

$$s_0^* \approx r - \frac{k_* r \alpha \sin(\varphi_0 + \alpha)}{r + k_* \cos(\varphi_0 + \alpha)} ; \qquad \Psi_0 \approx \frac{r\alpha}{r + k_* \cos(\varphi_0 + \alpha)} . \tag{18}$$

It follows from (18) that

$$\Delta_0 = r + k_* \cos(\varphi_0 + \alpha) - k_* \alpha \sin(\varphi_0 + \alpha) + \text{terms } (k_*^2, \alpha^2) .$$

which means that when $r + k_* \cos(\varphi_0 + \alpha) > 0$, $\Delta_0 > 0$ will remain in effect.

The singularity produced by the zero $s = s_0^*, \psi = \Psi_0$ in the denominator of the kernel component n = 0 depends on the instantaneous blade position φ_0. The relationship is relatively complicated, because values of r and φ_0 exist(3) for which $s_0^* > R_0$ or $s_0^* < R_i$. This means that the singularity disappears for short time intervals (that is for short φ_0-intervals). This is because s and r can only /290

(3) For $\varphi_0 + \alpha \approx 3\pi/2$ and $r \lesssim R_0$ as well as for $\varphi_0 + \alpha \approx \pi/2$ and $r \gtrsim R_i$.

take on values between R_i and R_0.

The solution of integral equations containing such kernel singularities becomes very complicated and laborious. This is especially true if several singularities of this type occur and if they are of various types.

We will find that such singularities are also contained in the kernel part of the blade $n = N - 1$. In this case, an exact analysis is even more necessary than for the blade n = 0, because the value of $k_* \Psi_{N-1}$ in Equation (11) is considerably larger than that of $k_* \Psi_0$ in Equation (13) or (18).

Therefore, it seems appropriate to carry out the analysis of the part $(u_Q^{(0)} + u_L^{(0)})$ using the assumption $k_* = 0$ based on no reverse flow along the blade when k_* is small. In this case the kernel singularity is given in the simple form (17). From Equations (6), (8), (14) and (15) we obtain the following lucid representation:

$$\begin{aligned}
\text{Part } (u_Q^{(0)} + u_L^{(0)}) = {} & \frac{1}{4\pi} \int_{R_i}^{R_0} \int_{-\alpha}^{\alpha} \Bigg\{ [r^2 + s^2 - 2\, r\, s \cos\chi]^{-3/2} \left[\frac{\partial \Gamma(s, \varphi_0 + \alpha + \chi)}{\partial s} (r\, s \cos\chi - s^2) - \right. \\
& \left. - \frac{\partial \Gamma(s, \varphi_0 + \alpha + \chi)}{\partial \varphi_0} r \sin\chi \right] - [(r-s)^2 + r^2 \chi^2]^{-3/2} \left(1 + \frac{3}{2} \frac{\chi^2 r (r-s)}{(r-s)^2 + r^2 \chi^2}\right) \times \\
& \times \left[\frac{\partial \Gamma(s, \varphi_0 + \alpha + \chi)}{\partial s} \left(r(r-s) - (r-s)^2 - \frac{1}{2} r^2 \chi^2 \right) - \frac{\partial \Gamma(s, \varphi_0 + \alpha + \chi)}{\partial \varphi_0} r \chi \right] \Bigg\} d\chi\, ds + \\
& + \frac{1}{4\pi} \int_{R_i}^{R_0} \Bigg\{ \int_{-\alpha}^{\alpha} [(r-s)^2 + r^2 \chi^2]^{-3/2} \left(1 + \frac{3}{2} \frac{\chi^2 r (r-s)}{(r-s)^2 + r^2 \chi^2}\right) \times \\
& \times \left[\frac{\partial \Gamma(s, \varphi_0 + \alpha + \chi)}{\partial s} \left(r(r-s) - (r-s)^2 - \frac{1}{2} r^2 \chi^2 \right) - \frac{\partial \Gamma(s, \varphi_0 + \alpha + \chi)}{\partial \varphi_0} r \chi \right] d\chi - \\
& - \left(\frac{2}{r-s} + \frac{1}{r} \ln \frac{|r-s|}{2 r \alpha} \right) \frac{\partial \Gamma(s, \varphi_0 + \alpha)}{\partial s} - \frac{2}{r^2} \ln \frac{|r-s|}{2 r \alpha} \frac{\partial^2 \Gamma(s, \varphi_0 + \alpha)}{\partial \varphi_0^2} \Bigg\} ds - \\
& - \frac{1}{4\pi} \int_{R_i}^{R_0} \Gamma(s, \varphi_0 + \alpha) \frac{ds}{r\, s} + \frac{1}{2\pi} \int_{R_i}^{R_0} \frac{\partial^3 \Gamma(s, \varphi_0 + \alpha)}{\partial s\, \partial \varphi_0^2} (r - s) \left[\ln \frac{|r-s|}{2 r \alpha} - 1 \right] \frac{ds}{r^2} + \\
& + \frac{1}{2\pi} \int_{R_i}^{R_0} \left(\frac{\partial \Gamma(s, \varphi_0 + \alpha)}{\partial s} + \frac{1}{2 s} \Gamma(s, \varphi_0 + \alpha) \right) \frac{ds}{r - s}.
\end{aligned} \tag{19}$$

β

In formula (19), the integrand of the first double integral is continuous and vanishes at the point $\chi = 0$ and s = r. Using the integral formulas (74) to (81) from Section 7, we also find that the integrand of the second integral with respect to s takes on the following value for s = r

$$\frac{1}{4\pi}\left(\frac{1}{2r}\frac{\partial\Gamma(r,\varphi_0+\alpha)}{\partial r}+\frac{2}{r^2}\frac{\partial^2\Gamma(r,\varphi_0+\alpha)}{\partial\varphi_0^2}\right)+O(\alpha^2)$$

This means that it is also continuous. The remaining terms are partially transformed by means of partial integration, so that the singular kernel part in the last line of (19) becomes clear. We have assumed that

$$\Gamma(R_1,\varphi_0)=\Gamma(R_0,\varphi_0)=0 \tag{20}$$

[See Section 7; Formulas (82), (83)].

b)

Leading blade $n = N-1$, longitudinal vortex.

We consider the Case II according to Equation (10), because in the Case I the integrand will be continuous anyway.

The integrand of the part $(u_L^{(N-1)})$ in the Case II becomes singular for the specified target point $r, \varphi+\alpha(r)$ when $s = s_{N-1}^*$ and $\psi = \Psi_{N-1}$ according to Equation System (11). In the vicinity of this point, the denominator of the part $(u_L^{(N-1)})$ has the form (14). It is only necessary to replace s_0^* by s_{N-1}^*, Ψ_0 by Ψ_{N-1} and $(\varphi+\Psi_0)$ /291

by $\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)$. We also find the following for the numerator

$$N = -\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)}{\partial\varphi_0}\left[\chi\left(s_{N-1}^*+k_*\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)-\right.\right.$$
$$\left.\left.-\chi^2 k_*\sin\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)\right)\right]. \tag{21}$$

If we assume that at the considered target point $r, \varphi_0+\alpha$ there will be no anomalous concentration of free vortex elements caused by vortex deformation, then the values $s = s_{N-1}^*$ and $\psi = \Psi_{N-1}$ will describe an isolated singularity of the denominator. This means that σ as well as χ must vanish $(\psi = \Psi_{N-1}+\chi;\ s = s_{N-1}^*+\sigma)$. Since the main part of the denominator has the following form for small σ and χ

$$\left[s^*_{N-1}\sin\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)\chi-\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)\sigma\right]^2+ + \left[s^*_{N-1}\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)\chi+\sin\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)\sigma+k_*\chi\right]^2$$

then the following determinant must satisfy the condition

$$s^*_{N-1}+k_*\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)>0 \tag{22}$$

if the zero point $\sigma=0$ and $\chi=0$ is to become possible. The relationship (22) is not important for the further transformations.

Equation system (11) can be solved for small k_* values (i.e., without reverse flow at target points) using a linearized approximation, if products of the type $(s^*_{N-1}-r)\left(\Psi_{N-1}-\frac{2\pi}{N}-\alpha\right)$ are first ignored. We then obtain

$$s^*_{N-1}\approx r-\frac{k_*\,r\left(\frac{2\pi}{N}+\alpha\right)\sin(\varphi_0+\alpha)}{r+k_*\cos(\varphi_0+\alpha)};\quad \Psi_{N-1}\approx\frac{2\pi}{N}+\alpha-\frac{k_*\left(\frac{2\pi}{N}+\alpha\right)\cos(\varphi_0+\alpha)}{r+k_*\cos(\varphi_0+\alpha)}. \tag{23}$$

The approximate solution (23) is of course less accurate than the corresponding relationship (18) for the blade n = 0. Nevertheless, (23) can be improved by means of an iteration. Since because of $\left(\alpha+\frac{2\pi}{N}\right)^2\gg\alpha^2$ the angle Ψ_{N-1} is considerably larger than Ψ_0, it is no longer permissible to ignore k_* (different from the case of the blade n = 0) when the part $(u_L^{(N-1)})$ is transformed or is set equal to zero.

Using the abbreviations

$$A_{N-1}=s^{*2}_{N-1}+k_*^2+2\,k_*\,s^*_{N-1}\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right),$$
$$B_{N-1}=k_*\sin\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right);$$
$$\Delta^2_{N-1}=\left(s^*_{N-1}+k_*\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)\right)^2=A_{N-1}-B^2_{N-1}$$

we obtain a formula which is similar to Equation (19):

Part

$$(u_L^{(N-1)})=\frac{1}{4\pi}\int_{R_i}^{R_0}\int_{-\Psi_{N-1}}^{-\alpha}\left[\left(r\cos(\varphi_0+\alpha)-s\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)\right)^2+ +\left(r\sin(\varphi_0+\alpha)-s\sin\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)-k_*(\Psi_{N-1}+\chi)\right)^2\right]^{-3/2}\times \tag{24}$$

$$\times\left[r\sin(\varphi_0+\alpha)\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)-r\cos(\varphi_0+\alpha)\sin\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)-\right.$$

$$\left.-k_*(\Psi_{N-1}+\chi)\cos\left(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)\right]\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)}{\partial\varphi_0}d\chi\,ds-$$

$$-\frac{1}{4\pi}\int_{R_i}^{R_0}\left\{\int_{-\alpha}^{\alpha}[\sigma^2+2\sigma\chi B_{N-1}+\chi^2A_{N-1}]^{-3/2}\left(1+\frac{3}{2}\frac{\chi^3s_{N-1}B_{N-1}+\sigma\chi^2(s_{N-1}-2\Delta_{N-1})}{\sigma^2+2\sigma\chi B_{N-1}+\chi^2A_{N-1}}\right)\times\right.$$

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$$\times(\chi\Delta_{N-1}-\chi^2B_{N-1})\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi\right)}{\partial\varphi_0}d\chi+$$

$$+\frac{2B_{N-1}}{\sqrt{A_{N-1}}}\left(\frac{1}{\Delta_{N-1}}\frac{1}{\sigma}-\frac{1}{A_{N-1}}\left(1-\frac{3}{2}\frac{s_{N-1}\Delta_{N-1}}{A_{N-1}}\right)\ln\frac{|\sigma|\Delta_{N-1}}{2\alpha A_{N-1}}\right)\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0}+\qquad(24)$$

$$\left.+\frac{2\Delta_{N-1}}{\sqrt{A_{N-1}}^3}\ln\frac{|\sigma|\Delta_{N-1}}{2\alpha A_{N-1}}\frac{\partial^2\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0^2}\right\}ds+$$

$$+\frac{1}{2\pi}\int_{R_i}^{R_0}\left\{\frac{B_{N-1}}{\sqrt{A_{N-1}}^3}\left(1-\frac{3}{2}\frac{s_{N-1}\Delta_{N-1}}{A_{N-1}}\right)\frac{\partial^2\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0\,\partial s}-\right.$$

$$\left.-\frac{\Delta_{N-1}}{\sqrt{A_{N-1}}^3}\frac{\partial^3\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0^2\,\partial s}\right\}\left[\ln\frac{|\sigma|\Delta_{N-1}}{2\alpha A_{N-1}}-1\right]\sigma\,ds-$$

$$-\frac{1}{2\pi}\frac{B_{N-1}}{\sqrt{A_{N-1}}}\frac{1}{\Delta_{N-1}}\int_{R_i}^{R_0}\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0}\frac{ds}{s_{N-1}^*-s};\qquad(\sigma=s-s_{N-1}^*).$$

In formula (24), the first double integral has a continuous integrand. Using the integral formulas (74) to (81) from Section 7, we also find that the integrand of the second integral with respect to s takes on the following value for $s=s_{N-1}^*$

$$+\frac{1}{2\pi}\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0}\left[\frac{3B_{N-1}^3-4A_{N-1}B_{N-1}}{\Delta_{N-1}^2\sqrt{A_{N-1}}^3}+\frac{s_{N-1}^*B_{N-1}(3{,}5A_{N-1}-4B_{N-1}^2)}{\Delta_{N-1}\sqrt{A_{N-1}}^5}\right]-$$

$$-\frac{1}{2\pi}\frac{\partial^2\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0^2}\frac{A_{N-1}-2B_{N-1}^2}{\Delta_{N-1}\sqrt{A_{N-1}}^3}+O(\alpha^2)$$

The last term in Equation (24) contains the only singular kernal part [see also formulas (82), (83), Section 7].

c)

Leading blade $n = N - 1$, rolled-up transverse vortex.

The part of the velocity component $(u_Q^{(N-1)})$ to be investigated consists of a rolled-up tip vortex and a hub vortex. Both have a similar structure. We will therefore discuss only one integrand and will drop the superscripts (0) and (i). The singularity in the integrand of part $(u_Q^{(N-1)})$ is given by the equation system (7). The values of r_0 and ψ_0 must be calculated from it for prescribed s_{N-1} and φ_0.

We will develop the denominator D and the numerator N of the part $(u_Q^{(N-1)})$ in the vicinity of this point. We will therefore set

$$\psi = \psi_0 + \chi, \quad r = r_0 + \varrho, \quad \alpha(r) = \alpha_0 + \alpha_0' \varrho$$

and consider only third order terms in the denominator. In the numerator we will only consider second order terms in χ and ϱ. Considering (7) we find the following after some elementary transformations

$$\begin{aligned} D = \Big\{ & \varrho^2 (1 + r_0^2 \alpha_0'^2) + 2 \varrho \chi \Big[s_{N-1} \sin\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) - k_* \sin(\varphi_0 + \alpha_0) - \\ & - s_{N-1} r_0 \alpha_0' \cos\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) - k_* r_0 \alpha_0' \cos(\varphi_0 + \alpha_0) \Big] + \\ & + \chi^2 \Big[s_{N-1}^2 + k_*^2 + 2 k_* s_{N-1} \cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi_0\Big)\Big] - \chi^3 k_* s_{N-1} \sin\Big(\varphi_0 - \frac{2\pi}{N} + \psi_0\Big) + \\ & + \varrho^3 r_0 \alpha_0'^2 + \varrho \chi^2 s_{N-1} \Big[\cos\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) + r_0 \alpha_0' \sin\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big)\Big] - 2 \varrho^2 \chi \alpha_0' \times \\ & \times \Big[s_{N-1} \cos\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) + k_* \cos(\varphi_0 + \alpha_0) + s_{N-1} \frac{r_0}{2} \alpha_0' \sin\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) - \\ & - k_* \frac{r_0}{2} \alpha_0' \sin(\varphi_0 + \alpha) \Big]\Big\}^{3/2} \end{aligned} \tag{25}$$

and except for Γ_m (sign of the tip vortex) /293

$$\begin{aligned} N = & - \varrho \Big[s_{N-1} \cos\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) + k_* \cos(\varphi_0 + \alpha_0)\Big] + \varrho \chi s_{N-1} \sin\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) + \\ & + \frac{1}{2} \chi^2 s_{N-1} \Big[s_{N-1} + k_* \cos\Big(\varphi_0 - \frac{2\pi}{N} + \psi_0\Big)\Big] - \alpha_0' \varrho (r_0 + \varrho) \Big[s_{N-1} \sin\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) - \\ & - k^* \sin(\varphi_0 + \alpha)\Big] + \frac{1}{2} \varrho^2 r_0 \alpha_0'^2 \Big[s_{N-1} \cos\Big(\psi_0 - \frac{2\pi}{N} - \alpha_0\Big) + k_* \cos(\varphi_0 + \alpha_0)\Big] - \\ & - \varrho \chi s_{N-1} r_0 \alpha_0' \cos\Big(\psi_0 - \alpha_0 - \frac{2\pi}{N}\Big). \end{aligned} \tag{26}$$

We will now restrict ourselves to small k_* values as before (i.e., without any reverse flow, that is $s_{N-1} + k_* \cos(\varphi_0 + \alpha_0) > 0$). This means that equation system (7) can be linearized with respect to $(r_0 - s_{N-1})$ and $\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right)$. The approximate solution then becomes

$$r_0 \approx s_{N-1} + \frac{k_* \left(\alpha_0 + \frac{2\pi}{N}\right) s_{N-1} \sin(\varphi_0 + \alpha_0)}{s_{N-1} + k_* \cos(\varphi_0 + \alpha_0)}; \qquad \psi_0 \approx \frac{\left(\alpha_0 + \frac{2\pi}{N}\right) s_{N-1}}{s_{N-1} + k_* \cos(\varphi_0 + \alpha_0)} \tag{7}$$

We will use the following abbreviations:

$$\begin{aligned}
A &= s_{N-1}^2 + k_*^2 + 2 k_* s_{N-1} \cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0\right); \qquad C = (1 + r_0^2 \alpha_0'^2) \\
B &= s_{N-1} \sin\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right) - k_* \sin(\varphi_0 + \alpha_0) - s_{N-1} r_0 \alpha_0' \cos\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right) - \\
&\quad - k_* r_0 \alpha_0' \cos(\varphi_0 + \alpha_0); \\
A C - B^2 &= \Big[s_{N-1} \cos\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right) + k_* \cos(\varphi_0 + \alpha_0) + \\
&\quad + s_{N-1} r_0 \alpha_0' \sin\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right) - k_* r_0 \alpha_0' \sin(\varphi_0 + \alpha_0) \Big]^2; \\
P_1 &= s_{N-1} \cos\left(\psi_0 + \frac{2\pi}{N} - \alpha_0\right) + k_* \cos(\varphi_0 + \alpha_0); \\
P_2 &= s_{N-1} \sin\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right) - k_* \sin(\varphi_0 + \alpha_0); \\
P_3 &= \cos\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right) + r_0 \alpha_0' \sin\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right); \\
P_4 &= \sin\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right) - r_0 \alpha_0' \cos\left(\psi_0 - \frac{2\pi}{N} - \alpha_0\right).
\end{aligned} \tag{28}$$

The value $A C - B^2$ is always different from zero and positive. This is true for the condition, not connected with any physical condition, that the rolled-up transverse vortices of the blade $n = N - 1$ do not touch the 3/4 line of the target blade but intersect it (Figure 8). The direction vector of the rolled-up transverse vortex is given by

$$ds_Q = \left[- s_{N-1} \sin\left(\varphi_0 + \psi - \frac{2\pi}{N}\right) e_y + s_{N-1} \cos\left(\varphi_0 + \psi - \frac{2\pi}{N}\right) e_z + k_* e_x \right] d\psi$$

and for the 3/4 line at the blade n = 0 it is given by

$$ds_r = [\cos(\varphi_0 + \alpha)\, e_y - r \alpha' \sin(\varphi_0 + \alpha)\, e_y + \sin(\varphi_0 + \alpha)\, e_z + r \alpha' \cos(\varphi_0 + \alpha)\, e_z]\, dr,$$

with

$$\alpha' = \frac{d\alpha}{dr}.$$

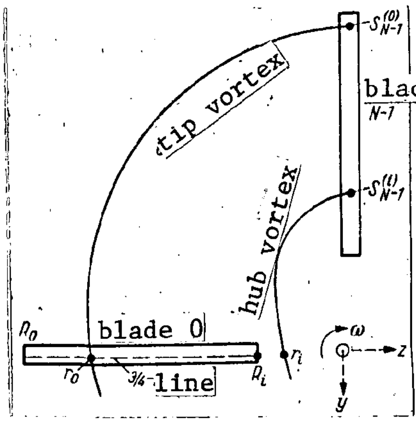

Fig. 8. Geometry of the tip and hub vortices of the blade N - 1 at the target blade.

As can be immediately calculated, we then have

$$(ds_r \times ds_Q)^2 = A\,C - B^2,$$

and the statement made above is therefore proven. Considering formulas (25), (26) and (28) we then obtain the desired equation: /294

$$\begin{aligned}
\text{Part } (u_Q^{(N-1)}) = {} & \frac{1}{4\pi} \int_{-\psi_0}^{-\alpha_0} \Gamma_m\left(\varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right) \left[\left(r\cos(\varphi_0+\alpha) - s_{N-1}\cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right)\right)^2 + \right. \\
& \left. + \left(r\sin(\varphi_0+\alpha) - s_{N-1}\sin\left(\varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right) - k_*(\psi_0+\chi)\right)^2\right]^{-3/2} \times \\
& \times \left[s_{N-1}^2 - r\,s_{N-1}\cos\left(\psi_0 + \chi - \frac{2\pi}{N} - \alpha\right) - k_*\,r\cos(\varphi_0+\alpha) + k_*\,s_{N-1}\cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right) + \right. \\
& \left. + k_*\,s_{N-1}(\psi_0+\chi)\sin\left(\varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right)\right] d\chi + \\
& + \left\{\frac{1}{4\pi} \int_{-\alpha_0}^{\alpha_0} \Gamma_m\left(\varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right) [A\chi^2 + 2B\chi\varrho + C\varrho^2]^{-3/2} \times \right. \\
& \times \left[1 + \frac{3}{2}\, \frac{\chi^3 k_* s_{N-1}\sin\left(\varphi_0 - \frac{2\pi}{N} + \psi_0\right) - \varrho^3 r_0 \alpha_0'^2 - \varrho\chi^2 s_{N-1} P_3 + 2\varrho^2\chi\alpha_0'\left(P_1 + \frac{r_0}{2}\alpha_0' P_2\right)}{A\chi^2 + 2B\chi\varrho + C\varrho^2}\right] \times \qquad (29) \\
& \times \left[-\varrho\sqrt{AC - B^2} + \varrho\chi\, s_{N-1} P_4 - \alpha_0'\varrho^2\left(P_2 - \frac{1}{2} r_0 \alpha_0' P_1\right) + \frac{1}{2}\chi^2 s_{N-1} \times \right. \\
& \left. \times \left(s_{N-1} + k_*\cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0\right)\right)\right] d\chi + \frac{1}{2\pi}\Gamma_m\left(\varphi_0 - \frac{2\pi}{N} + \psi_0\right) \times \\
& \left. \times \left[\sqrt{\frac{A}{AC - B^2}}\,\frac{1}{\varrho} + \frac{1}{2}\,\frac{1}{\sqrt{A}^3}\left(s_{N-1}^2 + s_{N-1}k_*\cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0\right)\right) \ln\frac{|\varrho|\sqrt{AC - B^2}}{2A\alpha}\right]\right\} - \\
& - \left[\frac{1}{2\pi}\sqrt{\frac{A}{AC - B^2}}\,\frac{r - r_0}{b_*^2 + (r - r_0)^2} + \frac{1}{4\pi}\,\frac{1}{\sqrt{A}^3}\left(s_{N-1}^2 + s_{N-1}k_*\cos\left(\varphi_0 - \frac{2\pi}{N} + \psi_0\right)\right) \times \right. \\
& \left. \times \ln\frac{\sqrt{b_*^2 + (r - r_0)^2}\sqrt{AC - B^2}}{2A\alpha}\right] \Gamma_m\left(\varphi_0 - \frac{2\pi}{N} + \psi_0\right).
\end{aligned}$$

Some explanation of formula (29) is required. The integrand of the first integral is apparently continuous. Using the integral formulas (74) to (81) from Section 7, we also find that the expression in the rounded brackets {} takes on the following value for $\varrho = 0$:

$$+\frac{1}{2\pi}\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi_0\right)\left[s_{N-1}P_3\left(\frac{B^2}{\sqrt{AC-B^2}^3}+\frac{1}{2}\frac{1}{\sqrt{AC-B^2}}\right)\frac{1}{\sqrt{A}}-\frac{s_{N-1}P_4B}{(AC-B^2)\sqrt{A}}-\right.$$

$$-\alpha_0'\left(P_2-\frac{1}{2}r_0\alpha_0'P_1\right)\frac{\sqrt{A}}{AC-B^2}+2\alpha_0'\left(P_1+\frac{r_0}{2}\alpha_0'P_2\right)\frac{B\sqrt{A}}{\sqrt{AC-B^2}^3}+\frac{r_0\alpha_0'\sqrt{A}^3}{\sqrt{AC-B^2}^3}+$$

$$+\frac{1}{2}\left(s_{N-1}^2+s_{N-1}k_*\cos\left(\varphi_0-\frac{2\pi}{N}+\psi_0\right)\right)\frac{2B^2-AC}{(AC-B^2)\sqrt{A}^3}+k_*s_{N-1}\sin\left(\varphi_0-\frac{2\pi}{N}+\psi_0\right)\times$$

$$\left.\times\left(\frac{BC}{\sqrt{AC-B^2}^3}\frac{1}{\sqrt{A}}+\frac{B}{\sqrt{AC-B^2}}\frac{0{,}5}{\sqrt{A}^3}\right)\right]+\frac{1}{2\pi}\frac{\partial\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi_0\right)}{\partial\varphi_0}\frac{B}{\sqrt{AC-B^2}\sqrt{A}}.$$

In the last terms of formula (29), the singular expressions $\frac{1}{\varrho}$ and $\ln|\varrho|$ respectively, which are singular according to the potential theory, have been replaced by $\frac{\varrho}{b_*^2+\varrho^2}$ $\ln\sqrt{b_*^2+\varrho^2}$ where $\varrho=r-r_0$. This is based on the fact that a rolled-up free transverse vortex behaves like a RANKINE vortex in a viscous flow and a finite (even though small) core radius, and does not have the behavior of a single vortex filament of potential theory. Such a velocity profile can be quite qell approximated by

$\varrho\,[b_*^2+\varrho^2]^{-1}$ with b_* as the core radius. (30)

Equation (30) is very suited for further treatment.

The part of Equation (29) corresponding to expression (30) gives a good description of the velocity field, when the target blade n = 0 penetrates through the rolled-up tip and hub vortices of the preceding blade $n=N-1$. /295

If the core radius is small (for example $b_*\approx 0{,}05\,R_0$), then the function (30) becomes discontinuous almost in a step-like manner. However, we must consider that according to Equation (27), depending on the instantaneous angular position φ_0 of the blade, it is possible to have value $r_0\leqq R_0$ as well as $r_0>R_0$ and corresponding to this $r_i\geqq R_i$ and $r_i<R_i$. Since the blade target points are restricted to the range $R_i\leqq r\leqq R_0$, this discontinuous, step-like course of the function (30) only exists for $r_i\geqq R_i$ and for $r_0\leqq R_0$. Only then do the rolled-up tip and hub vortices of the blade $n=N-1$ intersect the blade n = 0 (Figure 8). These properties must be considered in the solution of the integral equation derived from the flow boundary condition (12).

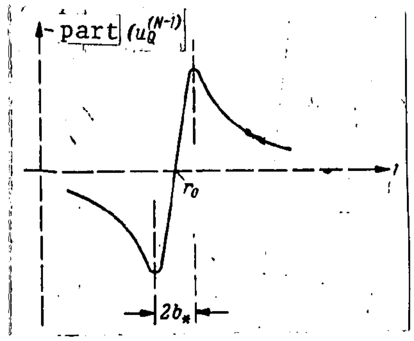

Fig. 9. Induced axial velocity for penetration of a tip vortex.

4. SOME INTEGRAL EQUATIONS

a)

We consider the integral equation

$$\frac{r - r_0(\varphi_0, \alpha_0)}{b_*^2 + [r - r_0(\varphi_0, \alpha_0)]^2} =$$

$$\frac{1}{2\pi}\int_{R_i}^{R_0}\left[\frac{\partial \gamma(s, \varphi_0 + \alpha_0)}{\partial s} + \frac{1}{2s}\gamma(s, \varphi_0 + \alpha_0)\right]\frac{ds}{r - s} \qquad \left(R_i \leq {r \atop s} \leq R_0,\; 0 \leq \varphi_0 \leq 2\pi\right) \tag{31}$$

r_0 can take on values which are larger than R_0. Using the transformation $b_* = 1/2\,(R_0 - R_i)\,b$ and

$$s = \frac{1}{2}(R_0 + R_i) - \frac{1}{2}(R_0 - R_i)\cos\tau\,; \quad r = \frac{1}{2}(R_0 + R_i) - \frac{1}{2}(R_0 - R_i)\cos\vartheta\,;$$

$$r_0 = \frac{1}{2}(R_0 + R_i) - \frac{1}{2}(R_0 - R_i)\cos\vartheta_0\,,\ (r_0 \leq R_0)\,; \quad r_0 = \frac{1}{2}(R_0 + R_i) + \frac{1}{2}(R_0 - R_i)\cosh\vartheta_0^*\,,\ (r_0 \geq R_0)$$

Equation (31) takes on the following form:

$$-2\,\frac{\cos\vartheta {-\cos\vartheta_0 \atop +\cosh\vartheta_0^*}}{b^2 + \left(\cos\vartheta {-\cos\vartheta_0 \atop +\cosh\vartheta_0^*}\right)^2} = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\partial \gamma(\tau, \varphi_0 + \alpha_0)}{\partial \tau} + \frac{1}{2}\,\frac{\sin\tau\cdot\gamma(\tau, \varphi_0 + \alpha_0)}{\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau}\right)\frac{d\tau}{\cos\tau - \cos\vartheta}\,, \tag{32}$$

$$\left(0 \leq {\vartheta \atop \tau} \leq \pi\,;\quad \vartheta_0 \leq \pi\,;\quad \vartheta_0^* \geq 0\right).$$

In Equation (32), we have $-\cos\vartheta_0$ for $r_0 \leq R_0$ and $\cosh\vartheta_0^*$ for $r_0 \geq R_0$. At the transition point $\vartheta_0 = \pi$ and $\vartheta_0^* = 0$ both equations agree.

Using the integral formula (64) proven in Section 7, we first find the following solution for the integral equation (32)

$$\frac{\partial \gamma(\tau, \varphi_0 + \alpha_0)}{\partial \tau} + \frac{1}{2}\,\frac{\sin\tau\cdot\gamma(\tau, \varphi_0 + \alpha_0)}{\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau} = b\sqrt{2}\,\frac{\sqrt{Q_1} + \sin^2\tau}{b^2 + (\cos\tau - \Xi)^2}\left[1 - \Xi^2 + b^2 + \sqrt{Q_1}\right]^{-1/2} + b\,\zeta(\varphi_0, \alpha_0)\,;$$

(ζ first undetermined)

$$\Xi = \cos\vartheta_0 \text{ for } r_0 \leq R_0\,; \quad \Xi = -\cosh\vartheta_0^* \text{ for } r_0 \geq R_0\,. \tag{33}$$

The values ϑ_0 and ϑ_0^* depend on the instantaneous angular position φ_0 just like r_0. Equation (33) can be looked upon as a linear ordinary differential equation for the desired function γ, as far as its dependence on τ is concerned. It can be immediately integrated using the known formula. The unknown function ζ (ζ is a constant with respect to the variable τ) as well as the integration constant which occurs for the integration ofthhe differential equation (33) can be determined from the necessary condition (20); i.e., $\gamma = 0$ for $\tau = 0$ and $\tau = \pi$.

After an elementary calculation, the following solution of the integral equation (31) and (32), respectively, is found:

$$\gamma(\tau, \varphi_0) = \frac{b\left[\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau\right]^{-1/2}}{\left[1 - \Xi^2 + b^2 + \sqrt{Q_1}\right]^{1/2}} \left\{ \int_0^\tau \frac{\left[\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau'\right]^{1/2}}{b^2 + (\cos\tau' - \Xi)^2} \sqrt{2}\left(\sqrt{Q_1} + \sin^2\tau'\right) d\tau' - \right.$$

$$\left. - \int_0^\tau \left[\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau'\right]^{1/2} d\tau' \left(\int_0^\pi \left[\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau'\right]^{1/2} d\tau' \right)^{-1} \int_0^\pi \frac{\left[\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau'\right]^{1/2}}{b^2 + (\cos\tau' - \Xi)^2} \sqrt{2}\left(\sqrt{Q_1} + \sin^2\tau'\right) d\tau' \right\}.$$

where

$$\Xi = \cos\vartheta_0 \text{ for } r \leqq R_0, \quad \Xi = -\cosh\vartheta_0^* \text{ for } r_0 \geqq R_0. \tag{34}$$

Abbreviation: /296

$$Q_1 \equiv (\Xi^2 - 1 - b^2)^2 + 4\,\Xi^2 b^2.$$

$f(\varphi_0, \alpha_0)$ is an arbitrary continuous function.

In a similar way, the integral equation corresponding to (31) can be solved, if r_0 is replaced by $r_i(\varphi_0, \alpha_0)$. Then we have $\Xi = \cos\vartheta_0$ for $r_i \geqq R_i$ and $\Xi = +\cosh\vartheta_0^*$ with $\vartheta_0^* \geqq 0$ for $r_i \leqq R_i$.

It seems appropriate to determine the order of magnitude of the solution (34) obtained.

After this, when the formula (34) is used, b_* must be looked upon as the core radius of the rolled-up transverse vortex. Measurements [4] have shown, that the core radius of rolled-up tip vortices of a rotor lies in the range 0.04 R_0 and 0.07 R_0. Together with $b_* = 1/2\,(R_0 - R_i)\,b$ this results in $b^2 \approx 10^{-2}$.

Since as a rule $(R_0 - R_i)^2/(R_0 + R_i)^2 \lesssim 1/2$, we also have the following good approximation:

$$\int_0^{\tau} \sqrt{\frac{R_0+R_i}{R_0-R_i}-\cos\tau'}\,d\tau' \approx \sqrt{\frac{R_0+R_i}{R_0-R_i}}\left(\tau-\frac{1}{2}\frac{R_0-R_i}{R_0-R_i}\sin\tau\right). \tag{35}$$

We will distinguish several cases when estimating the order of magnitude of the solution (34) for small values of b.

First of all we have $\sin^2\vartheta_0 \gg b^2$. Then using $\Xi = \cos\vartheta_0$ we have:

$$\begin{aligned}\gamma(\tau,\varphi_0) \approx b\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau\right)^{-1/2}&\left\{\int_0^{\tau}\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau'\right)^{1/2}\frac{\left(\sin\vartheta_0+\frac{\sin^2\tau'}{\sin\vartheta_0}\right)d\tau'}{b^2+(\cos\tau'-\cos\vartheta_0)^2}-\right.\\ &-\int_0^{\tau}\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau'\right)^{1/2}d\tau'\left(\int_0^{\pi}\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau'\right)d\tau'\right)^{-1}\times\\ &\left.\times\int_0^{\pi}\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau'\right)^{1/2}\frac{\left(\sin\vartheta_0+\frac{\sin^2\tau'}{\sin\vartheta_0}\right)d\tau'}{b^2+(\cos\tau'-\cos\vartheta_0)^2}\right\}.\end{aligned} \tag{36}$$

The dominant integrand in Equation (36) apparently has a steep maximum when $\tau' \approx \vartheta_0$ and to a good degree of approximation $(\varepsilon^2 \ll 1)$ we have:

$$\begin{aligned}&\int_0^{\pi}\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau'\right)^{1/2}\frac{\left(\sin\vartheta_0+\frac{\sin^2\tau'}{\sin\vartheta_0}\right)d\tau'}{b^2+(\cos\tau'-\cos\vartheta_0)^2}\approx\int_{\vartheta_0-\varepsilon}^{\vartheta_0+\varepsilon}(\ldots)\,d\tau'\approx\\ &\approx 2\sin\vartheta_0\left(\frac{R_0+R_i}{R_0-R_i}-\cos\vartheta_0\right)^{1/2}\int_{\vartheta_0-\varepsilon}^{\vartheta_0+\varepsilon}\frac{d\tau'}{b^2+\sin^2\vartheta_0(\tau'-\vartheta_0)^2}=\frac{4}{b}\left(\frac{R_0+R_i}{R_0-R_i}-\cos\vartheta_0\right)^{1/2}\arctan\frac{\varepsilon\sin\vartheta_0}{b}.\end{aligned} \tag{37}$$

Using (35) and (37) we then find the following estimation from Equation (36)

$$\gamma \approx \frac{\left(\frac{R_0+R_i}{R_0-R_i}-\cos\vartheta_0\right)^{1/2}}{\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau\right)^{1/2}}\,4\arctan\frac{\varepsilon\sin\vartheta_0}{b}\begin{cases}1-\frac{1}{\pi}\left(\tau-\frac{1}{2}\frac{R_0-R_i}{R_0+R_i}\sin\tau\right) & \text{for} \\ -\frac{1}{\pi}\left(\tau-\frac{1}{2}\frac{R_0-R_i}{R_0+R_i}\sin\tau\right) & \text{for}\end{cases} \quad (\sin^2\vartheta_0 \gg b^2) \tag{38}$$

Formula (38) shows that $\gamma \sim O(1)$ also holds for $b \to 0$. In addition (as can be expcoted from the structure of the integral equation (31) we find that γ is positive for $r > r_0$ and negative for $r < r_0$. This is clear if we consider the velocity field induced by a rolled-up tip vortex (Figure 10).

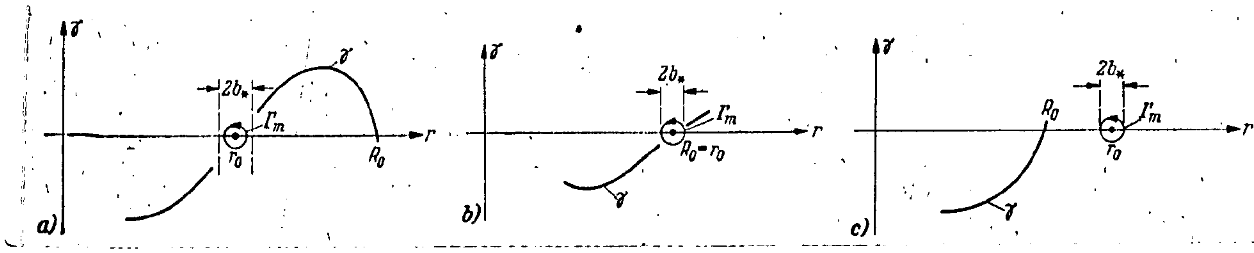

Fig. 10. Various positions of the tip vortex of blade N - 1 at the target blade. The circulation fraction γ produced by tip vortex N - 1 at blade 0.

Let us now assume that $\vartheta_0 = \pi$ or $\vartheta_0^* = 0$. Then using $\Xi = -1$: /297

$$\gamma \approx \frac{b}{\sqrt{b + \frac{1}{2} b^2}} \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau\right)^{-1/2} \left\{ \int_0^\tau \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau'\right)^{1/2} \frac{(2b + \sin^2\tau')\, d\tau'}{b^2 + (\cos\tau' + 1)^2} - \right.$$
$$\left. - \int_0^\tau \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau'\right)^{1/2} d\tau' \left(\int_0^\pi \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau'\right)^{1/2} d\tau' \right)^{-1} \int_0^\pi \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau'\right)^{1/2} \frac{(\sin^2\tau' + 2b)\, d\tau'}{b^2 + (\cos\tau' + 1)^2} \right\}. \tag{39}$$

Considering relationship (see formulas (65), Section 7)

$$\frac{1}{\pi} \int_0^\pi \left(1 - \frac{1}{2} \frac{R_0 - R_t}{R_0 + R_t} \cos\tau\right) \frac{(\sin^2\tau + 2b)\, d\tau}{b^2 + (\cos\tau + 1)^2} \approx \frac{1}{\sqrt{b}} \left(2 + \frac{R_0 - R_t}{R_0 + R_t}\right) - 1 - \frac{R_0 - R_t}{R_0 + R_t} \tag{40'}$$

as well as Equation (35), we obtain the following estimation from (39)

$$\gamma \approx \sqrt{b} \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau\right)^{-1/2} \left\{ \int_0^\tau \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau'\right)^{1/2} \frac{(2b + \sin^2\tau')\, d\tau'}{b^2 + (\cos\tau' + 1)^2} - \right.$$
$$\left. - \left(\tau - \frac{1}{2} \frac{R_0 - R_t}{R_0 + R_t} \sin\tau\right) \frac{1}{\sqrt{b}} \left(2 + \frac{R_0 - R_t}{R_0 + R_t} - \sqrt{b} - \sqrt{b}\, \frac{R_0 - R_t}{R_0 + R_t}\right) \right\} \approx \gamma \tag{40}$$

for

$$\tau < \vartheta_0, \quad \vartheta_0 = \pi, \quad \vartheta_0^* = 0.$$

From the representation (40) we find $\gamma \sim O(1)$ for $b \to 0$. It is understandable that γ is negative, because in (40) we have $r_0 = R_0$ that is, $r < r_0$. Finally, we consider the case $\sinh^2 \vartheta_0^* \gg b^2$ and $\sinh \vartheta_0^* \approx \cosh \vartheta_0^*$. This means that the rolled-up transverse vortex lies at a certain distance outside of the blade (Figure 10).

From (34) we obtain the following for $b \to 0$:

$$\gamma \approx \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau\right)^{-1/2} \left\{ \int_0^\tau \left(\frac{R_0 + R_t}{R_0 - R_t} - \cos\tau'\right)^{1/2} \frac{(\sinh^2 \vartheta_0^* + \sin^2\tau')\, d\tau'}{(\cosh \vartheta_0^* + \cos\tau')^2} - \right. \tag{41}$$

$$-\frac{1}{\pi}\left(\tau-\frac{1}{2}\frac{R_0-R_i}{R_0+R_i}\sin\tau\right)\int_0^{\pi}\left(\frac{R_0+R_i}{R_0-R_i}-\cos\tau'\right)^{1/2}\frac{(\sinh^2\vartheta_0^*+\sin^2\tau')\,d\tau'}{(\cosh\vartheta_0^*+\cos\tau')^2}\Bigg\}. \tag{41}$$

As ϑ_0^* increases (that is as the distance between the rolled-up transverse vortex and the blade increases), from (41) we find that $\gamma\to 0$, as should be the case. This is because $(\sinh^2\vartheta_0^*+\sin^2\tau')(\cosh\vartheta_0^*+\cos\tau')^{-2}$ becomes independent of τ', and therefore the integrals with respect to τ' can be approximated by (35).

This then concludes the discussion of the exact solution (34) of the integral equation (31). The integral equation corresponding to (31) can be investigated in exactly the same way where $r_i(\varphi_0,\alpha_0)$ is replaced by $r_0(\varphi_0,\alpha_0)$.

The derivative $\partial\gamma/\partial\varphi_0$ of the solution (34) with respect to the angle φ_0, which plays the role of a parameter, remains continuous at the transition point $r_0=R_0$, $\vartheta_0=\pi$ and $\vartheta_0^*=0$. This is true because $dr_0/d\varphi_0$ is continuous according to equation (27) and also we have the following abbreviation in (34)

$r_0=\frac{1}{2}(R_0+R_i)-\frac{1}{2}(R_0-R_i)\,\Xi$, it follows that $\frac{d\Xi}{d\varphi_0}=-\frac{2}{R_0-R_i}\frac{dr_0}{d\varphi_0}$;

This means that $\partial\gamma/\partial\varphi_0$ remains continuous in the range $0\leqq\varphi_0\leqq 2\pi$.

b)

We will consider the integral equation $\left(R_i\leqq {r \atop s}\leqq R_0;\ 0\leqq\varphi_0\leqq 2\pi\right)$:

$$\begin{aligned}
&F(r,\varphi_0,\alpha)+\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi_0\right)f(\varphi_0,\alpha_0)\frac{r-r_0(\varphi_0,\alpha_0)}{b_*^2+[r-r_0(\varphi_0,\alpha_0)]^2}=\\
&=\frac{1}{2\pi}\int_{R_i}^{R_0}\left[\frac{\partial\Gamma(s,\varphi_0+\alpha_0)}{\partial s}+\frac{1}{2s}\Gamma(s,\varphi_0+\alpha_0)\right]\frac{ds}{r-s}+\\
&+\frac{1}{2\pi}f_*(r,\varphi_0)\int_{R_i}^{R_0}\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0}\frac{ds}{s-s_{N-1}^*(r,\varphi_0)}+q(r,\varphi_0)\,\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi_0\right)+\\
&+\frac{1}{2\pi}\int_{\chi_A}^{\chi_B}\Gamma_m\left(\varphi_0-\frac{2\pi}{N}+\psi_0+\chi\right)Q(r,\varphi_0,\chi)\,d\chi+\frac{1}{2\pi}\int_{R_i}^{R_0}\Bigg\{\Gamma(s,\varphi_0+\theta_1)\,K_1(r,s,\varphi_0)+\\
&+\frac{\partial\Gamma(s,\varphi_0+\theta_2)}{\partial s}K_2(r,s,\varphi_0)+\frac{\partial\Gamma(s,\varphi_0+\theta_3)}{\partial\varphi_0}K_3(r,s,\varphi_0)+\\
&+\int_{(\chi)}\left[\frac{\partial\Gamma(s,\varphi_0+\theta_4+\chi)}{\partial s}K_4(r,s,\varphi_0,\chi)+\frac{\partial\Gamma(s,\varphi_0+\theta_5+\chi)}{\partial\varphi_0}K_5(r,s,\varphi_0,\chi)\right]d\chi\Bigg\}\,ds.
\end{aligned} \tag{42}$$

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The structure of Equation (42) must be explained.

First of all (42) has the integral equation (31) as a part. In addition, there is a partially singular core component which has the new form $[s - s^*_{N-1}(r, \varphi_0)]^{-1}$. The expression $s^*_{N-1}(r, \varphi_0)$ is given by Equation (23). This means that values $R_i \leqq s^*_{N-1} \leqq R_0$ as well as $s^*_{N-1} < R_i$, $s^*_{N-1} > R_0$ are possible. The core component mentioned above is not singular for the latter case.

Γ_m is a special value of the desired function Γ. When applied to the rotor flow, we are dealing with the circulation of the rolled-up tip and hub vortex. We will again discuss the definition of Γ_m.

K_1 to K_4 as well as Q are continuous core components of the integral Equation (42) for all variables. $\int\limits_{(\chi)}$ is a symbolic notation for integrals with respect to χ for various possible limits. Of course their existence is assumed. ϑ_1 to ϑ_5 are phase displacements with respect to φ_0. $q(r, \varphi_0)$, $f_*(r, \varphi_0)$, $f(\varphi_0, \alpha_0)$, $F(r, \varphi_0, \alpha_0)$ are prescribed continuous functions.

For the solution of Equation (42) we first carry out the transformation

$$s = \frac{1}{2}(R_0 + R_i) - \frac{1}{2}(R_0 - R_i)\cos\tau \qquad (\varphi_\alpha = \varphi_0 - \alpha_0)$$

and use the trial solution:

$$\Gamma(s, \varphi_0) \equiv \Gamma(\tau, \varphi_0) = \Gamma_m\left(\varphi_\alpha - \frac{2\pi}{N} + \psi_0\right) f(\varphi_\alpha, \alpha_0)\,\gamma(\tau, \varphi_\alpha) + \Gamma_*(\tau, \varphi_0). \tag{43}$$

Let $\gamma(\tau, \varphi_0)$ be the solution of the integral Equation (31). The function γ is therefore known and can be immediately determined from Equation (34). In the case of rotor flow, γ characterizes the influence of a transverse vortex which is penetrated by the rotor blade on the blade circulation. Since $\gamma(0, \varphi_0) = 0, \gamma(\pi, \varphi_0) = 0$, γ can be written in the form of a Fourier sine series (even if there is a relatively steep function for a corresponding number L of terms):

$$\gamma(\tau, \varphi_0) = \sum_{\lambda=1}^{L} \gamma_\lambda(\varphi_0) \sin\lambda\tau. \tag{44}$$

The other trial function Γ_* is essentially determined by the remainder of the flow field in which the blade moves. It is also necessary for Γ_* to vanish for $\tau = 0$ and $\tau = \pi$ and will be assumed to have the form of a Fourier series:

$$\Gamma_*(\tau,\varphi_0)=\sum_{\mu=-M_0}^{M_0}\sum_{\nu=1}^{M}a_{\mu\nu}\,e^{i\mu\varphi_0}\sin\nu\tau\,.$$
$$(0\leqq\tau\leqq\pi\,,\quad 0\leqq\varphi_0\leqq 2\pi)\,. \tag{45}$$

We will use the idea that the rolling-up process of the free vortices occurs according to the following law. The vortex material in the range $r>\frac{R_0+R_i}{2}$ makes up the tip vortex and the material in the range $r<\frac{R_0+R_i}{2}$ makes up the hub vortex. Then we have (4)

$$\Gamma_m(\varphi_0)\approx\Gamma\left(\frac{R_0+R_i}{2},\varphi_0\right)=\Gamma\left(\frac{\pi}{2},\varphi_0\right).$$

If we approximately set $\psi_0\approx(2\pi)/N$, then from (43) we have (α_0 ignored)

$$\Gamma_m(\varphi_0)\approx\Gamma_m(\varphi_0)\,f(\varphi_0,\alpha_0)\,\gamma\left(\frac{\pi}{2},\varphi_0\right)+\Gamma_*\left(\frac{\pi}{2},\varphi_0\right),$$

or

$$\Gamma_m(\varphi_0)=\frac{\Gamma_*\left(\frac{\pi}{2},\varphi_0\right)}{1-f(\varphi_0,\alpha_0)\,\gamma\left(\frac{\pi}{2},\varphi_0\right)}\,. \tag{46}$$

According to our estimation (38) we have $\gamma(\pi/2,\varphi_0)<0$ and because we can also assume that $f>0$ according to Equation (29), the denominator in (46) will not vanish.

Before substituting the trial solution (43) to (46) into the integral Equation (42), we will first give some formulas for the rational calculation of the singular core components in Equation (42). /299

First of all, we will perform the following transformation:

$$\begin{aligned}
s_{N-1}^*&=\frac{R_0+R_i}{2}-\frac{R_0-R_i}{2}\cos\xi\,, && \text{for } R_i\leqq s_{N-1}^*\leqq R_0\,;\\
s_{N-1}^*&=\frac{R_0+R_i}{2}\pm\frac{R_0-R_i}{2}\cosh\xi^*\,; && \begin{matrix}+\\-\end{matrix}\left|\begin{matrix}\text{for } s_{N-1}^*\geqq R_0\\ \text{for } s_{N-1}^*\leqq R_i\,;\end{matrix}\right. \quad (\xi_*\geqq 0)\\
r&=\frac{R_0+R_i}{2}-\frac{R_0-R_i}{2}\cos\theta\,;
\end{aligned} \tag{47}$$

(4) Basically, another τ value can be used to determine $\pi/2$ in place of Γ_m.

Considering the integral equations (66) to (68) from Section 7, we then find

$$\frac{1}{2\pi}\int_{R_i}^{R_a}\left\{\frac{\partial\Gamma_*(s,\varphi_0+\alpha_0)}{\partial s}+\frac{1}{2s}\Gamma_*(s_0\varphi_0+\alpha_0)\right\}\frac{ds}{r-s}=\frac{1}{2\pi}\int_{R_i}^{R_a}\Gamma_*(s,\varphi+\alpha)\frac{ds}{2rs}+$$

$$+\frac{1}{2\pi}\int_{R_i}^{R_a}\left\{\frac{\partial\Gamma_*(s,\varphi_0+\alpha_0)}{\partial s}+\frac{1}{2r}\Gamma_*(s,\varphi_0+\alpha_0)\right\}\frac{ds}{r-s}=$$

$$=\sum_{\mu=-M_a}^{M_a}\sum_{\nu=1}^{M}a_{\mu,\nu}\,e^{i\mu(\varphi_0+\alpha_0)}\left\{\frac{1}{R_0-R_i}\frac{\sin\nu\vartheta}{\sin\vartheta}-\frac{1}{4\nu r(\vartheta)}\cos\nu\vartheta+\frac{1}{2\pi\nu}\int_0^{\pi}\frac{R_0-R_i}{4s(\tau)\,r(\vartheta)}\sin\nu\tau\sin\tau\,d\tau\right\}.$$

$$\left(R_i\leqq\frac{r}{s}\leqq R_0\right). \tag{48}$$

$$\frac{1}{2\pi}\int_{R_i}^{R_a}\frac{\partial\Gamma\left(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0}\frac{ds}{s-s^*_{N-1}(r,\varphi_0)}=\frac{1}{2}\sum_{\mu=-M_a}^{M_a}\sum_{\nu=1}^{M}a_{\mu\nu}\,i\,\mu\,e^{i\mu\left(\varphi_a-\frac{2\pi}{N}+\Psi_{N-1}\right)}\begin{pmatrix}\cos\nu\xi\\ e^{-\nu\xi^*}\\ (-1)^\nu e^{-\nu\xi^*}\end{pmatrix}+$$

$$+\frac{1}{2}\sum_{\mu=-M_a}^{M_a}\sum_{\nu=1}^{M}a_{\mu\nu}\sin\nu\frac{\pi}{2}\frac{\partial}{\partial\varphi_0}\left\{e^{i\mu\left(\varphi_a-\frac{4\pi}{N}+\Psi_{N-1}+\psi_0\right)}f\left(\varphi_a-\frac{2\pi}{N}+\Psi_{N-1},\alpha_0\right)\times\right.$$

$$\times\sum_{\lambda=1}^{L}\gamma_\lambda\left(\varphi_a-\frac{2\pi}{N}+\Psi_{N-1}\right)\left[1-f\left(\varphi_a-\frac{4\pi}{N}+\Psi_{N-1}+\psi_0\right)\gamma\left(\frac{\pi}{2},\varphi_a-\frac{4\pi}{N}+\Psi_{N-1}+\psi_0\right)\right]^{-1}\times$$

$$\times\begin{pmatrix}\cos\lambda\xi\\ e^{-\lambda\xi^*}\\ (-1)^\lambda e^{-\lambda\xi^*}\end{pmatrix}. \qquad (\varphi_a=\varphi_0-\alpha_0). \tag{49}$$

In Equation (49), the upper sign in the bracket expressions hold for $R_i\leqq s^*_{N-1}\leqq R_0$, the middle one holds for $s^*_{N-1}\leqq R_i$ and the lower one holds for $s^*_{N-1}\geqq R_0$.(5) In the case of the rotor flow we have for

$0\leqq\varphi_0+\alpha\leqq\pi$	and	$R_i\leqq s^*_{N-1}<R_0$	and	$s^*_{N-1}\leqq R_i,$	and for
$\pi\leqq\varphi_0+\alpha\leqq 2\pi$	and	$R_i<s^*_{N-1}\leqq R_0$	and	$s^*_{N-1}\geqq R_0.$	are possible.

[See Equation (23)]. The transformed variables ξ,ξ^* then lie in the following ranges according to formula (47):

$0\leqq\varphi_0+\alpha\leqq\pi\ :\ 0\leqq\xi\leqq\xi_1(\varphi_0)$	and	$0\leqq\xi^*\leqq\xi_1^*(\varphi_0)\,;$
$\pi\leqq\varphi_0+\alpha\leqq 2\pi\ :\ \xi_2(\varphi_0)\leqq\xi\leqq\pi$	and	$0\leqq\xi^*\leqq\xi_2^*(\varphi_0)\,;$

(5) The values in brackets transform continuously into each other at the transition points $\xi=0,\xi^*=0$ and $\xi=\pi,\xi^*=0$ in a continuous way. We have $s^*_{N-1}=s^*_{N-1}(r,\varphi_0)$. The derivative $\partial/\partial\varphi_0$ in (49) must only be carried out with respect to the explicit φ_0 values but not with respect to the phase displacements Ψ_{N-1} und ψ_0 which also depend on φ_0. (Because Equation (46) is also obtained with the assumption $\varphi_0\approx(2\pi)/N$, that is, independent of φ_0).

with

$$
\begin{aligned}
\xi_1(\varphi_0) &= \arccos\left(\frac{R_0+R_i-2\,s^*_{N-1}(R_0,\varphi_0)}{R_0-R_i}\right); \quad \xi_1(-\alpha)=\xi_1(\pi-\alpha)=\pi\,.\\
\xi_1^*(\varphi_0) &= \operatorname{arcosh}\left(\frac{R_0+R_i-2\,s^*_{N-1}(R_i,\varphi_0)}{R_0-R_i}\right); \quad \xi_1^*(-\alpha)=\xi_1^*(\pi-\alpha)=0\,.\\
\xi_2(\varphi_0) &= \arccos\left(\frac{R_0+R_i-2\,s^*_{N-1}(R_i,\varphi_0)}{R_0-R_i}\right); \quad \xi_2(-\alpha)=\xi_2(\pi-\alpha)=0\,.\\
\xi_2^*(\varphi_0) &= \operatorname{arcosh}\left(\frac{R_0+R_i-2\,s^*_{N-1}(R_0,\varphi_0)}{R_i-R_0}\right); \quad \xi_2^*(-\alpha)=\xi_2^*(\pi-\alpha)=0\,.
\end{aligned}
\tag{50}
$$

The trial solution (43) to (46) for the integral Equation (42) can be written /300 in the following clearer form, which depends linearly on the solution coefficients $a_{\mu\nu}$ which must be calculated:

$$
\Gamma(s,\varphi_0)\equiv\Gamma(\tau,\varphi_0)=\sum_{\mu=-M_0}^{M_0}\sum_{\nu=1}^{M}a_{\mu\nu}\,e^{i\mu\varphi_0}\left\{\sin\nu\tau+\frac{e^{-i\mu\left(\frac{2\pi}{N}-\psi_0+\alpha_0\right)}\sin\nu\frac{\pi}{2}f(\varphi_a,\alpha_0)\,\gamma(\tau,\varphi_a)}{1-f\left(\varphi_a-\frac{2\pi}{N}+\psi_0,\alpha_0\right)\gamma\left(\frac{\pi}{2},\varphi_a-\frac{2\pi}{N}+\psi_0\right)}\right\}
$$
$$
[a_{-\mu,\nu}=\bar{a}_{\mu\nu}]\,. \tag{51}
$$

Here $\gamma(\tau,\varphi_0)$ is the already known solution (34) of the integral equation (31). It characterizes the effect of the penetrated rolled-up transverse vortex (tip vortex) of the preceding blade on the circulation distribution over the blade, in the case of rotor flow. The trial solution given in the form (51) very clearly shows the separation of this solution. The remainder of the solution consists of a conventional $\sin\nu\tau$ series obtained from lifting line theory.

With (51), the integral equation (42) takes on the following form:

$$
\begin{aligned}
I\,g/(r,\varphi_0) &\equiv F(r,\varphi_0,\alpha)-\frac{1}{2\pi}\int_{R_i}^{R_0}\left[\frac{\partial\Gamma_*(s,\varphi_0+\alpha_0)}{\partial s}+\frac{1}{2s}\Gamma_*(s,\varphi_0+\alpha_0)\right]\frac{ds}{r-s}+\\
&+\frac{1}{2\pi}f_*(r,\varphi_0)\int_{R_i}^{R_0}\frac{\partial\Gamma\left(s;\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)}{\partial\varphi_0}\,\frac{ds}{s^*_{N-1}(r,\varphi_0)-s}-\\
&-\frac{q(r,\varphi_0)\,\Gamma_*\left(\frac{\pi}{2},\varphi_0-\frac{2\pi}{N}+\psi_0\right)}{1-f\left(\varphi_0-\frac{2\pi}{N}+\psi_0,\alpha_0\right)\gamma\left(\frac{\pi}{2},\varphi_0-\frac{2\pi}{N}+\psi_0\right)}-\frac{1}{2\pi}\int_{\chi_A}^{\chi_E}Q(r,\varphi_0,\chi)\times
\end{aligned}
\tag{52}
$$

$$\times \frac{\Gamma_*\left(\frac{\pi}{2}, \varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right) d\chi}{1 - f\left(\varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi, \alpha_0\right) \gamma\left(\frac{\pi}{2}, \varphi_0 - \frac{2\pi}{N} + \psi_0 + \chi\right)} - \frac{1}{2\pi} \int_{R_i}^{R_0} \left\{ \Gamma(s, \varphi_0 + \theta_1) K_1(r, s, \varphi_0) + \right.$$

$$+ \frac{\partial \Gamma(s, \varphi_0 + \theta_2)}{\partial s} K_2(r, s, \varphi_0) + \frac{\partial \Gamma(s, \varphi_0 + \theta_3)}{\partial \varphi_0} K_3(r, s, \varphi_0) + \tag{52}$$

$$\left. + \int_{(\chi)} \left[\frac{\partial \Gamma(s, \varphi_0 + \theta_4 + \chi)}{\partial s} K_4(r, s, \varphi_0, \chi) + \frac{\partial \Gamma(s, \varphi_0 + \theta_5 + \chi)}{\partial \varphi_0} K_5(r, s, \varphi_0, \chi) \right] d\chi \right\} ds = 0.$$

$$(0 \leqq \varphi_0 \leqq 2\pi; \quad R_i \leqq r \leqq R_0).$$

It is most appropriate to solve (52) according to the method of the smallest square error, that is, using the condition

$$\sum_{l=1}^{2L_0} \sum_{j=1}^{L} \{I\, g/(r_j, \varphi_{0l})\}^2 = \text{Minimum}. \tag{53}$$

Here r_j, φ_{0l} are selected target points. As a rule $\varphi_{0l} = \frac{l\pi}{L_0}$ and $r_j = \frac{1}{2}(R_0 + R_i) - \frac{1}{2}(R_0 - R_i)\cos\frac{j\pi}{L+1}$. The condition (53) and the trial solution (51) leads to a linear system of equations for solving the $2M_0M$ solution coefficients $a_{\mu\nu}$. In the special case L = M and $L_0 = M_0$, the minimum vanishes. As a rule, we will select the approximation $L > M$ in order to have a uniform approximation. Sometimes, we will also select $L_0 > M_0$.

The two singular integrands in Equation (52) are evaluated using formulas (48) and (49). The other integrands, which are all continuous are solved using the conventional quadrature methods. The fact that depending on the target point r_j, φ_{0l}, we have either $R_i \leqq s^*_{N-1} \leqq R_0$ or $s^*_{N-1} < R_i$ or $s^*_{N-1} > R_0$, and therefore Equation (49) is represented in three ways, does not influence the solution method in any way. It is only necessary to determine how the quantity s^*_{N-1} depends on the selected target point r_j, φ_{0l}. This can be done using Equation (23).

5. THE INTEGRAL EQUATION OF THE FLOW BOUNDARY CONDITION /301

In Sections 2 and 3 we discussed the velocities induced at the rotor blade using the vortex model under discussion. If we substitute this representation (3) to (6), (8) to (10) as well as (19), (24) and (29) in the flow boundary condition (12) at the rotor blade, then we obtain an integral equation of the general type (42) as discussed in Section 4b.

It is not necessary to write down the individual complicated core components again. They can be directly taken from the formulas mentioned above. The continuous cores must be summarized in a clearer notation, as was done in Equation (42). However, we should notice that there is a difference between Equation (42) and the integral equation derived from (12). This is due to the fact that two characteristic terms appear produced by rolled-up transverse vortices (that is, tip vortices and hub vortices). This changes nothing in the method of solution. The trial solution (43) must only be extended as follows (with $\varphi_a = \varphi_0 - 1/2\,\alpha_0 - 1/2\,\alpha_i$)

$$\Gamma(s,\varphi_0) \equiv \Gamma(\tau,\varphi_0) = \Gamma_m\left(\varphi_a - \frac{2\pi}{N} + \psi_0^{(0)}\right) f_0(\varphi_a,\alpha_0)\,\gamma_0(\tau,\varphi_a) + \Gamma_m\left(\varphi_a - \frac{2\pi}{N} + \psi_0^{(i)}\right) f_i(\varphi_a,\alpha_i)\,\gamma_i(\tau,\varphi_a) + \Gamma_*(\tau,\varphi_0), \tag{54}$$

where γ_0 and γ_i are solutions of the integral equation [see (31), (34)]:

$$\frac{r - r_0(\varphi_0,\alpha_0)}{b_*^2 + [r - r_0(\varphi_0,\alpha_0)]^2} = \frac{1}{2\pi}\int_{R_i}^{R_0}\left[\frac{\partial\gamma_0(s,\varphi_0)}{\partial s} + \frac{1}{2s}\gamma_0(s,\varphi_0)\right]\frac{ds}{r-s}; \qquad \left(\alpha \approx \frac{1}{2}\alpha_0 + \frac{1}{2}\alpha_i\right) \tag{55}$$

$$-\frac{r - r_i(\varphi_0,\alpha_i)}{b_*^2 + [r - r_i(\varphi_0,\alpha_i)]^2} = \frac{1}{2\pi}\int_{R_i}^{R_0}\left[\frac{\partial\gamma_i(s,\varphi_0)}{\partial s} + \frac{1}{2s}\gamma_i(s,\varphi_0)\right]\frac{ds}{r-s}. \tag{55i}$$

Instead of Equation (46) we now have the following with $\psi_0^{(0)} \approx (2\pi)/N$ und $\psi_0^{(i)} \approx (2\pi)/N$:

$$\Gamma_m(\varphi_0) = \Gamma_*\left(\frac{\pi}{2},\varphi_0\right)\left[1 - f_0(\varphi_0,\alpha_0)\gamma_0\left(\frac{\pi}{2},\varphi_0\right) - f_i(\varphi_0,\alpha_i)\,\gamma_i\left(\frac{\pi}{2},\varphi_0\right)\right]^{-1}. \tag{56}$$

Since $\gamma_0(\pi/2,\varphi_0) < 0$ and $\gamma_i(\pi/2,\varphi_0) < 0$, the denominator in (56) is always positive.

Except for the small additions mentioned, the solution of the boundary condition integral equation (12) is exactly the same as for (42) and (52). This means we do not have to discuss them further.

6. CALCULATION OF THE BLADE FORCES

The axial K_z and circumferential K_φ forces which occur at the rotor blades (per length in the longitudinal direction) are determined using the KUTTA-JOUKOWSKI theorem [1], [2]. The forces in other directions can be composed of the two known force components.

As is well known, the velocities induced by the vortex system at the lifting line of the target blade $(x = 0,\ y = r\cos\varphi_0,\ z = r\sin\varphi_0)$ are required in the KUTTA-JOUKOWSKI theorem.

In Sections 2 and 3 of the present paper, we investigated velocities induced by the vortex system of the rotor along the 3/4 line $((x = 0, y = r\cos(\varphi_0 + \alpha), z = r\sin(\varphi_0 + \alpha))$ of the target blade. We must test whether these representations can also be used for calculating the induced velocities at the location of the lifting line (x = 0, $y = r\cos\varphi_0, z = r\sin\varphi_0)$. We must also determine the required modifications, if any.

A review of the results of Sections 2 and 3 shows that:

In v_r we must set $\alpha = 0$ and the sum n = 0 must be dropped.

In $\sum_{n=1}^{N-2} v_Q^{(n)}$ and $\sum_{n=1}^{N-2} v_L^{(n)}$ according to Equations (5) and (9) we must only set $\alpha = 0$.

In $r_Q^{(N-1)}$ and $v_L^{(\bar{N}-1)}$ in Equations (4), (10), (24) and (29) we must set $\alpha = \alpha_0 = \alpha_i = \alpha_0' = \alpha_i' = 0$. This is not true for the integration limits and $\varkappa$-values which remain unchanged. In addition, in Equations (7) and (11), (21) to (23) and (25) to (28) we must set $\alpha = \alpha_0 = \alpha_i = \alpha_0' = \alpha_i' = 0$.

The above statements also hold for the second integrals of $v_Q^{(0)}$ and $v_L^{(0)}$ from formula (6) and (8). Only the first two integrals of these induced velocities, which were already investigated in Section 3a and were abbreviated by the part $(u_Q^{(0)} + u_L^{(0)})$ along the 3/4 line of the target blade, must be modified at the position of the lifting line from the representation given in Section 3a. Consequently, we will also use the simplifying assumption $k_* = 0$. The necessity for changing this representation is also due to the fact that these induced velocity components become singular if the coordinates $x = 0, y = r\cos\varphi_0, z = r\sin\varphi_0$ of the lifting line are simply substituted [1], [2]. The expressions for determining the forces according to the KUTTA-JOUKOWSKI theorem avoid any singularity and are obtained from the lifting surface theory. The angle φ_0 of the lifting vortex line is not used. /302
Instead we use

$$\varphi_0 + \lim_{\chi \to 0} \chi$$

where χ is the angular coordinate in the blade chord direction. $\chi_v(r)$ refers to the leading edge and $\chi_H(r)$ to the trailing edge (Figure 7).

If integrands are found for $\chi \to 0$ which are integrable as CAUCHY principle values, ln singularities or continuous functions, we can immediately set $\chi = 0$ in the corresponding components. The critical terms, for which divergent integrals with respect to s would be produced for $\chi \to 0$, can be interpreted using the following integral formula [1], [2]:

$$\lim_{\chi\to 0}\int\limits_{R_i}^{R_0}\frac{ds}{\sqrt{(r-s)^2+r^2\chi^2}}=\lim_{\chi\to 0}\ln\frac{4\,(R_0-r)\,(r-R_i)}{r^2\chi^2}=2\ln\frac{4\sqrt{(R_0-r)\,(r-R_i)}}{r\,[\chi_H(r)-\chi_V(r)]}+2\ln 2+1\,. \tag{57}$$

We will have to refer to the literature for the basis of this method and the derivation of (57).

Using the method developed above, we first obtain the following from Equation (6) and (8) for $x=0, y=r\cos(\varphi_0+\chi), z=r\sin(\varphi_0+\chi)$ and $\chi\to 0$:

$$\text{Part }(u_Q^{(0)}+u_L^{(0)})=\frac{1}{4\pi}\lim_{\chi\to 0}\int\limits_{R_i}^{R_0}\int\limits_{0}^{2\alpha}[r^2+s^2-2\,r\,s\cos(\chi-\psi)]^{-3/2}\times$$
$$\times\left\{\frac{\partial\Gamma(s,\varphi_0+\psi)}{\partial s}\left(r\,s\cos(\chi-\psi)-s^2\right)+\frac{\partial\Gamma(s,\varphi_0+\psi)}{\partial\varphi_0}\,r\sin(\chi-\psi)\right\}d\psi\,ds\,. \tag{58}$$

A simple investigation of the integrals in the vicinity of the location s = r and $\psi=0$ shows that for part $(u_Q^{(0)})$, we may set $\chi=0$ directly. We find the following clear representation

$$\text{Part }(u_Q^{(0)})=\frac{1}{4\pi}\int\limits_{R_i}^{R_0}\int\limits_{0}^{2\alpha}\left\{[r^2+s^2-2\,r\,s\cos\psi]^{-3/2}\,\frac{\partial\Gamma(s,\varphi_0+\psi)}{\partial s}(r\,s\cos\psi-s^2)-\right.$$
$$\left.-\,[(r-s)^2+r\,s\,\psi^2]^{-3/2}\,\frac{\partial\Gamma(s,\varphi_0)}{\partial s}\,s\left(r-s-r\frac{\psi^2}{2}\right)\right\}d\psi\,ds+$$
$$+\frac{1}{4\pi}\int\limits_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0)}{\partial s}\left[\frac{2\alpha s}{\sqrt{4\alpha^2 r s+(r-s)^2}}\,\frac{1}{r-s}+\frac{\alpha}{\sqrt{4\alpha^2 r s+(r-s)^2}}-\right.$$
$$\left.-\frac{1}{2\sqrt{r s}}\ln\frac{2\alpha\sqrt{r s}+\sqrt{4\alpha^2 r s+(r-s)^2}}{|r-s|}\right]ds\,. \tag{59}$$

The integrand of the double integral in formula (59) is limited for s = r and $\psi=0$. By means of a series development with respect to ψ it can be shown that it has the value

$$\lim_{\psi\to 0}\lim_{s\to r}\left(\frac{\psi\,s\,(r-s)-\frac{1}{2}\psi^3\,r\,s}{\sqrt{(r-s)^2+r\,s\,\psi^{2}}^{3}}\right)\frac{\partial^2\Gamma(s,\varphi_0)}{\partial s\,\partial\varphi_0}=-\frac{1}{2\,r}\,\frac{\partial^2\Gamma(r,\varphi_0)}{\partial r\,\partial\varphi_0}$$

The integrand of the second simple integral over s (59) contains the known singularity of a CAUCHY principle value and that of a logarithm, which can be easily evaluated. For the part $(u_L^{(0)})$ and using the substitution $\psi-\chi=\eta$ we find the following representation for the limiting transition $\chi\to 0$:

$$\begin{aligned}\text{Part }(u_L^{(0)}) = & -\frac{1}{4\pi}\int\limits_{R_i}^{R_0}\int\limits_0^{2\alpha}\Big\{[r^2+s^2-2\,r\,s\cos\eta]^{-3/2}\frac{\partial\Gamma(s,\varphi_0+\eta)}{\partial\varphi_0}\,r\sin\eta - \\ & -[(r-s)^2+r\,s\,\eta^2]^{-3/2}\,r\,\eta\left(\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}+\eta\,\frac{\partial^2\Gamma(s,\varphi_0)}{\partial\varphi_0^2}\right)\Big\}\,d\eta\,ds + \\ & +\frac{1}{4\pi}\int\limits_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{s}\frac{ds}{\sqrt{4\alpha^2 r\,s+(r-s)^2}}+\frac{1}{4\pi}\int\limits_{R_i}^{R_0}\frac{\partial^2\Gamma(s,\varphi_0)}{\partial\varphi_0^2}\Big[\frac{2\alpha}{s\sqrt{4\alpha^2 r\,s+(r-s)^2}} - \\ & -\frac{1}{\sqrt{r\,s^3}}\ln\frac{2\alpha\sqrt{r\,s}+\sqrt{4\alpha^2 r\,s+(r-s)^2}}{|r-s|}\Big]\,ds-\frac{1}{4\pi}\lim_{\chi\to 0}\int\limits_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{r}\frac{ds}{\sqrt{\chi^2 r\,s+(r-s)^2}}.\end{aligned} \tag{60}$$

This last integral formula (60) is interpreted using Equation (57). We obtain /303

$$\begin{aligned}\lim_{\chi\to 0}\int\limits_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{s}\frac{ds}{\sqrt{\chi^2 r\,s+(r-s)^2}} = & \int\limits_{R_i}^{R_0}\left(\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{s}-\frac{\partial\Gamma(r,\varphi_0)}{\partial\varphi_0}\frac{1}{s}\right)\frac{ds}{|r-s|} + \\ & +\frac{1}{r}\frac{\partial\Gamma(r,\varphi_0)}{\partial\varphi_0}\left[2\ln\frac{4\sqrt{(R_0-r)\,(r-R_i)}}{r[\chi_H(r)-\chi_V(r)]}+2\ln 2+1\right].\end{aligned} \tag{61}$$

The integral on the right side of (61) has a discontinuous integrand at s = r. It has the value zero, which is its arithmetic mean value.

We must still determine the limiting value of the integrand for the double integral in formula (60) for s = r and $\eta = 0$. The following is obtained by means of a series development with respect to η:

$$\begin{aligned}\lim_{\eta\to 0}\lim_{s\to r} r\left[-\frac{1}{6}\frac{\eta^3}{\sqrt{(r-s)^2+r\,s\,\eta^2}^3}+\frac{1}{8}\frac{r\,s\,\eta^5}{\sqrt{(r-s)^2+r\,s\,\eta^2}^5}\right]\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0} + \\ +\frac{1}{2}\lim_{\eta\to 0}\lim_{s\to r} r\,\frac{\eta^3}{\sqrt{(r-s)^2+r\,s\,\eta^2}^3}\frac{\partial^2\Gamma(s,\varphi_0)}{\partial\varphi_0^2} = -\frac{1}{24}\frac{1}{r^2}\frac{\partial\Gamma(r,\varphi_0)}{\partial\varphi_0}+\frac{1}{2}\frac{1}{r^2}\frac{\partial^2\Gamma(r,\varphi_0)}{\partial\varphi_0^2}.\end{aligned}$$

This concludes the investigation of the induced velocities required for the force calculation according to the KUTTA-JOUKOWSKI theorem.

7. INTEGRAL FORMULAS

In the preceding sections, we used several important integral formulas which will be summarized and derived in the following.

a)

We consider the integral which can be evaluated using the residue method over the unit circle $Z = e^{i\tau}$ of the complex Z plane

$$I_n = \frac{1}{\pi}\int_0^{\pi} \frac{\cos n\tau\, d\tau}{(\cos\tau - \cos\vartheta)\,[b^2 + (\cos\tau - \Xi)^2]} =$$
$$= \frac{4}{\pi i}\oint_{(EK)} \frac{Z^{n-1}\, dZ}{\left(Z + \frac{1}{Z} - 2\cos\vartheta\right)\left(Z + \frac{1}{Z} - 2\,\Xi - 2\,i\,b\right)\left(Z + \frac{1}{Z} - 2\,\Xi + 2\,i\,b\right)}.$$

The zeroes in the denominator of the integrand are located at $Z_1 = e^{i\vartheta}$, $Z_2 = \Xi + i\,b + \sqrt{\Xi^2 - 1 - b^2 + 2\,i\,b\,\Xi}$, $Z_3 = \Xi + i\,b - \sqrt{\Xi^2 - 1 - b^2 + 2\,i\,b\,\Xi}$ as well as at the conjugate complex values.

We find

$$Z_1 \bar{Z}_1 = 1\,;\quad Z_2 Z_3 = 1\,;\quad Z_1 + \bar{Z}_1 = 2\cos\vartheta\,;$$
$$Z_2 + \frac{1}{Z_2} = Z_3 + \frac{1}{Z_3} = 2\,\Xi + 2\,i\,b\,.$$

For applications to rotor flow we must consider $b^2 \ll 1$. When evaluating the integral I_n and when discussing the zeroes Z_2, Z_3, we will cut the complex Z plane along the positive real axis in order to specify the principle values of the roots (6). An elementary series development and consideration of all the possibilities $|\Xi^2 - 1| \ll b^2$ with $\Xi = \pm 1$, or $|\Xi^2 - 1| \gg b^2$ with $\Xi > 1, \Xi < 1, \Xi < -1, \Xi > -1$ or $\Xi^2 - 1 = b^2$ with $\Xi \approx \pm(1 + b^2/2)$ shows that we always have $|Z_2| > 1$, that is $|Z_3| < 1$. Using the residue method we then obtain

$$I_n = \frac{1}{b^2 + (\cos\vartheta - \Xi)^2}\left[\frac{\sin n\vartheta}{\sin\vartheta} + \frac{i}{2\,b}\frac{\Xi - \cos\vartheta - i\,b}{\sqrt{(\Xi + i\,b)^2 - 1}}\,[\Xi + i\,b - \sqrt{(\Xi + i\,b)^2 - 1}]^n - \right.$$
$$\left. - \frac{i}{2\,b}\frac{\Xi - \cos\vartheta + i\,b}{\sqrt{(\Xi - i\,b)^2 - 1}}\,[\Xi - i\,b - \sqrt{(\Xi - i\,b)^2 - 1}\,]^n\right]. \tag{62}$$

In particular we find the following from formula (62):

$$I_0 = \frac{1}{b}\,\frac{(\Xi - \cos\vartheta)\sin\frac{\Phi}{2} + b\cos\frac{\Phi}{2}}{[b^2 + (\cos\vartheta - \Xi)^2]\sqrt[4]{Q_1}}\,; \tag{63}$$

$$I_2 = -\frac{2}{b}\,\frac{\sqrt[4]{Q_1}}{b^2 + (\cos\vartheta - \Xi)^2}\left[(\Xi - \cos\vartheta)\sin\frac{\Phi}{2} - b\cos\frac{\Phi}{2}\right] + \frac{1}{b}\,\frac{(\Xi - \cos\vartheta)\sin\frac{\Phi}{2} + b\cos\frac{\Phi}{2}}{[b^2 + (\cos\vartheta - \Xi)^2]\sqrt[4]{Q_1}}\,;$$

---

(6) The Z plane can also be cut along the negative real axis and the same result is obtained for I_n if the corresponding relationships are considered.

Using the notation: $\Xi^2 - 1 - b^2 + 2ib\Xi \equiv \sqrt{Q_1}\, e^{i\Phi}$ and the abbreviation $Q_1 = (\Xi^2 - 1 - b^2)^2 + 4\Xi^2 b^2$

The combination $\frac{\sqrt[4]{Q_1}}{\sin\frac{\Phi}{2}} I_0 + \frac{1}{2}\frac{I_0 - I_2}{\sqrt[4]{Q_1}\sin\frac{\Phi}{2}}$ then results in the desired integral re- /304

quired for the rotor flow:

$$\frac{b}{\sqrt{2}}\frac{1}{\pi}[1 - \Xi^2 + b^2 + \sqrt{Q_1}]^{-1/2} \int_0^\pi \frac{\sqrt{Q_1} + \sin^2\tau}{(\cos\tau - \cos\vartheta)[b^2 + (\cos\tau - \Xi)^2]} d\tau = \frac{\Xi - \cos\vartheta}{b^2 + (\cos\vartheta - \Xi)^2}; \qquad (64)$$

since

$$\sin\frac{\Phi}{2} = \frac{\sqrt{1 - \Xi^2 + b^2 + \sqrt{Q_1}}}{\sqrt{2}\sqrt[4]{Q_1}}; \qquad \left(0 < \frac{\Phi}{2} < \pi\right)$$

In a similar way we can prove the following integral formulas using the residue method

$$\frac{1}{\pi}\int_0^\pi \frac{2b + \sin^2\tau}{b^2 + (\cos\tau - 1)^2} d\tau = -1 + 2\frac{\left(1 + \frac{b}{2}\right)\cos\frac{\Phi_*}{2} + \sin\frac{\Phi_*}{2}}{\sqrt[4]{4b^2 + b^4}} \approx -1 + \frac{2}{\sqrt{b}} + \frac{1}{2}\sqrt{b} + O(b);$$

$$\frac{1}{\pi}\int_0^\pi \cos\tau \frac{2b^2 + \sin^2\tau}{b^2 + (\cos\tau - 1)^2} d\tau = 2 - \frac{(2 + 3b)\cos\frac{\Phi_*}{2} + (2 - 2b - b^2)\sin\frac{\Phi_*}{2}}{\sqrt[4]{4b^2 + b^4}} \approx$$

$$\approx 2 - \frac{2}{\sqrt{b}} - \frac{1}{2}\sqrt{b} + O(b). \qquad (65)$$

Here we have $2ib + b^2 = \sqrt{4b^2 + b^4}\, e^{i\Phi_*}$; $\tan\Phi_* = 2/b$.

Finally, we obtain the following with the residue method by integrating over the unit circle of the complex Z plane:

$$\frac{1}{\pi}\int_0^\pi \frac{\cos\nu\tau\, d\tau}{\cos\tau - \cos\vartheta} = \frac{\sin\nu\vartheta}{\sin\vartheta}; \qquad (0 \leq \vartheta \leq \pi) \qquad (66)$$

$$\frac{1}{\pi}\int_0^\pi \frac{\cos\nu\tau\, d\tau}{\cos\tau - \cosh\vartheta^*} = -\frac{e^{-\nu\vartheta^*}}{\sinh\vartheta^*}; \qquad (\vartheta^* > 0) \qquad (67)$$

$$\frac{1}{\pi}\int_0^\pi \frac{\cos\nu\tau\, d\tau}{\cos\tau + \cosh\vartheta^*} = (-1)^\nu \frac{e^{-\nu\vartheta^*}}{\sinh\vartheta^*}; \qquad (\vartheta^* > 0). \qquad (68)$$

b)

From the integral tables we are familiar with the three integrals[7]

[7] See, for example, footnote 9 on page 29 in [1].

$\int [Q_2]^{-3/2} \chi^m \, d\chi$ $\quad (m = 0, 1, 2)$

using the abbreviation $Q_2 = A\chi^2 + 2B\chi + C$.

By differentiating with respect to A, B or C we obtain the following integrals:

$$\int \frac{d\chi}{\sqrt{Q_2}^5} = \frac{1}{3}\frac{A\chi + B}{AC - B^2}\frac{1}{\sqrt{Q_2}^3} + \frac{2}{3}\frac{A\chi + B}{(AC - B^2)^2}\frac{A}{\sqrt{Q_2}}; \tag{69}$$

$$\int \frac{\chi\, d\chi}{\sqrt{Q_2}^5} = -\frac{1}{3}\frac{B\chi + C}{AC - B^2}\frac{1}{\sqrt{Q_2}^3} - \frac{2}{3}\frac{A\chi + B}{(AC - B^2)^2}\frac{B}{\sqrt{Q_2}}; \tag{70}$$

$$\int \frac{\chi^2\, d\chi}{\sqrt{Q_2}^5} = \frac{1}{3}\frac{A\chi + B}{AC - B^2}\frac{\chi^2}{\sqrt{Q_2}^3} + \frac{2}{3}\frac{B\chi + C}{(AC - B^2)^2}\frac{B}{\sqrt{Q_2}}; \tag{71}$$

$$\int \frac{\chi^3\, d\chi}{\sqrt{Q_2}^5} = -\frac{1}{3}\frac{B\chi + C}{AC - B^2}\frac{\chi^2}{\sqrt{Q_2}^3} - \frac{2}{3}\frac{B\chi + C}{(AC - B^2)^2}\frac{C}{\sqrt{Q_2}}; \tag{72}$$

$$\int \frac{\chi^4\, d\chi}{\sqrt{Q_2}^5} = \frac{1}{\sqrt{A}^5}\ln\left|A\chi + B + \sqrt{A}\sqrt{Q_2}\right| + \frac{2}{3}\frac{C\chi}{A(AC - B^2)}\frac{1}{\sqrt{Q_2}} + \frac{2}{3}\frac{(2AC - B^2)(2B^2\chi - AC\chi + BC)}{A^2(AC - B^2)^2\sqrt{Q_2}} + \frac{1}{3}\frac{\chi^2}{A(AC - B^2)}\frac{(2B^2 - AC)\chi + BC}{\sqrt{Q_2}^3} - \frac{2}{3}\frac{1}{A^2\sqrt{Q_2}}\frac{\sqrt{A}\sqrt{Q_2}\,\chi + A\chi^2 + B\chi + \frac{1}{2}C}{A\chi + B + \sqrt{A}\sqrt{Q_2}}. \tag{73}$$

If in the integral formulas above, we replace B by $B\sigma$ and C by $C\sigma^2$, we obtain the following important relationships using the abbreviation $Q_3 = A\chi^2 + 2B\chi\sigma + C\sigma^2$: /305

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\sigma\, d\chi}{\sqrt{Q_3}^3} - \frac{2\sqrt{A}}{\sigma(AC - B^2)}\right] = 0; \tag{74}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\chi\, d\chi}{\sqrt{Q_3}^3} + \frac{2B}{\sigma(AC - B^2)\sqrt{A}}\right] = 0; \tag{75}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\chi^2\, d\chi}{\sqrt{Q_3}^3} + \frac{1}{\sqrt{A}^3}\ln\frac{\sigma^2(AC - B^2)}{4A^2\alpha^2}\right] = \frac{2(2B^2 - AC)}{\sqrt{A}^3(AC - B^2)}; \tag{76}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\sigma^4\, d\chi}{\sqrt{Q_3}^5}\right] = \frac{4}{3}\frac{\sqrt{A}^3}{(AC - B^2)^2}; \tag{77}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\chi\sigma^3\, d\chi}{\sqrt{Q_3}^5}\right] = -\frac{4}{3}\frac{B\sqrt{A}}{(AC - B^2)^2}; \tag{78}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\chi^2\sigma^2\, d\chi}{\sqrt{Q_3}^5}\right] = \frac{4}{3}\frac{B^2}{(AC - B^2)^2\sqrt{A}} + \frac{2}{3}\frac{1}{(AC - B^2)\sqrt{A}}; \tag{79}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\sigma\chi^3\, d\chi}{\sqrt{Q_3}^5}\right] = -\frac{4}{3}\frac{BC}{(AC - B^2)^2\sqrt{A}} - \frac{2}{3}\frac{B}{(AC - B^2)\sqrt{A}^3}; \tag{80}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\chi^4\, d\chi}{\sqrt{Q_3}^5} + \frac{1}{\sqrt{A}^5}\ln\frac{\sigma^2(AC - B^2)}{4A^2\alpha^2}\right] = \frac{4}{3}\frac{2B^2}{(AC - B^2)\sqrt{A}^5} + \frac{4}{3}\frac{(2AC - B^2)(2B^2 - AC)}{(AC - B^2)^2\sqrt{A}^5}. \tag{81}$$

Assuming that $\Gamma(R_i) = \Gamma(R_0) = 0$ the following integral equations hold for continuously differentiable functions $\Gamma(s)$:

$$\frac{1}{2}\int_{R_i}^{R_0}\frac{d\Gamma(s)}{ds}\ln\frac{(s - r)^2}{R_0^2}\,ds = -\int_{R_i}^{R_0}\frac{\Gamma(s)}{s - r}\,ds; \tag{82}$$

$$\int_{R_i}^{R_a} \Gamma(s) \ln \frac{(s-r)^2}{R_0^2} ds = - \int_{R_i}^{R_a} \frac{d\Gamma(s)}{ds} \left[\ln \frac{(s-r)^2}{R_0^2} - 2 \right] (s-r) ds . \tag{83}$$

APPENDIX

8. THE ACOUSTIC PRESSURE FIELD OF A ROTOR

In many cases, the acoustic pressure field radiated by a rotor is of interest. Sometimes considerable noise is produced if a rotor blade penetrates the free tip or hub vortices. This is a process which we treated in this paper. The compressibility of the air must be considered in the calculation of the radiated acoustic far field [1]. Even in the case where the flow velocities at the rotor blade are still far enough from the velocity of sound so that an incompressible calculation can be carried out for the boundary conditions (8) (as was done in Sections 2 to 6 in the present paper). We will use the usual linearization assumptions of flow acoustics during the development of the corresponding formulas. This consists of a linear pressure-density relationship. We also ignore the squares or products of all velocities induced by the system of singularities of the rotor [1].

In this case, we then have the following linear relationship between the velocity potential of this singularity system (dipoles, sources) and the pressure field

$$\frac{1}{\varrho_0} p - \frac{1}{\varrho_0} p_0 = - \frac{\partial \Phi}{\partial t} - u_0 \frac{\partial \Phi}{\partial x} - w_0 \frac{\partial \Phi}{\partial z} = \omega \frac{\partial \Phi}{\partial \varphi_0} - u_0 \frac{\partial \Phi}{\partial x} - w_0 \frac{\partial \Phi}{\partial z} \tag{84}$$

where $\omega\, dt \doteq - d\varphi_0$. The velocity potential Φ then satisfies the general wave equation

$$\left(1 - \frac{u_0^2}{c_0^2}\right) \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \left(1 - \frac{w_0^2}{c_0^2}\right) \frac{\partial^2 \Phi}{\partial z^2} - 2 \frac{u_0 w_0}{c_0^2} \frac{\partial^2 \Phi}{\partial x \partial z} - \frac{2 u_0}{c_0^2} \frac{\partial^2 \Phi}{\partial x \partial t} - \frac{2 w_0}{c_0^2} \frac{\partial^2 \Phi}{\partial z \partial t} - \frac{1}{c_0^2} \frac{\partial^2 \Phi}{\partial t^2} = f(x, y, z, t), \tag{85}$$

where $f(x, y, z, t) = \sum_{(n)} Q(\mathbf{r}_n, t)\, \delta(\mathbf{r} - \mathbf{r}_n)$ only differs from zero at the location of the /306
source singularities. In Equation (85), c_0 is the speed of sound which is assumed to be constant over the entire flow field. u_0, w_0 are the incident velocities with respect to the helicopter in the x and z direction (Figure 1). The pressure field p also satisfies a wave equation of type (85) according to (84). However,

(8) As can be determined by a series development according to the reciprocal velocity of sound $1/c_0$, the formulas for the compressible and incompressible theory only differ by terms $\sim 1/c_0^2$. This means that the range of validity of the incompressible theory is exteneed for the near field (see [1], page 117, 119).

because of the inhomogeneous right side

$$-\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + w_0 \frac{\partial}{\partial z}\right) \varrho_0 \sum_{(n)} Q(r_n, t)\, \delta(r - r_n).$$

Equation (85) differs from the wave equations which usually occur in normal propeller theory by the fact that two principle incident flow velocities occur in the x and z directions. Therefore, we will transform (85) to a new coordinate system X, y, Z, i.e., we will set

$$\begin{aligned} x &= X\cos\lambda - Z\sin\lambda; & z &= X\sin\lambda + Z\cos\lambda \\ u_0 &= U\cos\lambda; & w_0 &= U\sin\lambda. \end{aligned} \tag{86}$$

Using the transformed potential

$$\Phi_1(X, y, Z, t) = \Phi_1(x\cos\lambda + z\sin\lambda, y, -x\sin\lambda + z\cos\lambda, t) \equiv \Phi(x, y, z, t) \tag{87}$$

Equation (85) then takes on the simpler form

$$\begin{aligned} &\left(1 - \frac{U^2}{c_0^2}\right)\frac{\partial^2 \Phi_1}{\partial X^2} + \frac{\partial^2 \Phi_1}{\partial y^2} + \frac{\partial^2 \Phi_1}{\partial Z^2} - \frac{2U}{c_0^2}\frac{\partial^2 \Phi_1}{\partial X\, \partial t} - \frac{1}{c_0^2}\frac{\partial^2 \Phi_1}{\partial t^2} = \\ &= f(X\cos\lambda - Z\sin\lambda, y, X\sin\lambda + Z\cos\lambda, t) \equiv F(X, y, Z, t), \end{aligned} \tag{88}$$

The solution of (88) is known and equals [1]

$$\Phi_1(X, y, Z, t) = -\frac{1}{4\pi}\iiint_{(\mathfrak{B})} \frac{1}{D} F\left(X', y', Z', t + \frac{U}{\beta^2 c_0^2}(X - X') - \frac{D}{\beta^2 c_0}\right) dX'\, dy'\, dZ'$$

where

$$\begin{aligned} D &= [(X - X')^2 + \beta^2 (y - y')^2 + (Z - Z')^2 \beta^2]^{1/2}; \\ \beta^2 &= 1 - U^2/c_0^2 = 1 - (u_0^2 + w_0^2)/c_0^2. \end{aligned} \tag{89}$$

In Equation (89), the integration must be extended over the range $(\mathfrak{B})$, in which F is different from zero. From (89) we also have

$$\Phi(x, y, z, t) = -\frac{1}{4\pi}\iiint_{(\mathfrak{B})} \frac{1}{D} f\left(x', y', z'; t + \frac{u_0}{\beta^2 c_0^2}(x - x') + \frac{w_0}{\beta^2 c_0^2}(z - z') - \frac{D}{\beta^2 c_0}\right) dx'\, dy'\, dz' \tag{90}$$

where $D = \left[(x - x')^2\left(1 - \frac{w_0^2}{c_0^2}\right) + (z - z')^2\left(1 - \frac{u_0^2}{c_0^2}\right) + \beta^2 (y - y')^2 + 2(x - x')(z - z')\frac{u_0 w_0}{c_0^2}\right]^{1/2}$,

or using cylindrical coordinates and $\omega\, dt = -d\varphi_0$:

$$\begin{aligned} \Phi(x, r, \varphi, \varphi_0) = -\frac{1}{4\pi}\iiint_{(\mathfrak{B})} \frac{1}{D} f\Big(x', r', \varphi', \varphi_0 &- \frac{\omega u_0}{\beta^2 c_0^2}(x - x') - \\ &- \frac{\omega w_0}{\beta^2 c_0^2}(r\sin\varphi - r'\sin\varphi') + \frac{\omega D}{\beta^2 c_0}\Big) r'\, dr'\, d\varphi'\, dx' \end{aligned} \tag{91}$$

where

$$D = \left[(x - x')^2\left(1 - \frac{w_0^2}{c_0^2}\right) + \left(1 - \frac{u_0^2}{c_0^2}\right)(r^2 + r'^2 - 2\,r\,r'\cos(\varphi - \varphi')) - \right.$$
$$\left. - \frac{w_0^2}{c_0^2}(r\cos\varphi - r'\cos\varphi')^2 + 2\frac{u_0\,w_0}{c_0^2}(x - x')(r\sin\varphi - r'\sin\varphi')\right]^{1/2}. \tag{91}$$

From Equation (91) we can now determine the basic solution Φ_0 for treating the rotor flow. This is the solution for a source element of intensity $q(s, \chi, \varphi_0)$ rotating at the angular velocity ω and which has the instantaneous position $x = k_1\chi,\ r = s,$ $\varphi = \varphi_0 + \chi$. In this special case, we have

$$f(x', r', \varphi', \varphi_0) = q(s, \chi, \varphi_0)\,\delta\,(x - k_1\chi)\frac{1}{r'}\,\delta\,(r' - s)\,\delta\,(\varphi' - \varphi_0 - \chi)$$

Using elementary calculations (see [1], page 110) we obtain the following from Equation (91): /307

$$\Phi_0(x, r, \varphi, \varphi_0) = -\frac{1}{4\pi}q\left(s, \chi, \varphi_0 - \frac{\omega\,u_0}{\beta^2 c_0^2}(x - k_1\chi) - \frac{\omega\,w_0}{\beta^2 c_0^2}(r\sin\varphi - s\sin\sigma_0) + \frac{\omega\,D_0}{\beta^2 c_0}\right)\times$$
$$\times\left[D_0\left(1 - \frac{w_0\,\omega\,s}{\beta^2 c_0^2}\cos\sigma_0\right) + \frac{\omega\,r\,s}{\beta^2 c_0}\left(1 - \frac{u_0^2}{c_0^2}\right)\sin(\varphi - \sigma_0) + \right.$$
$$\left. + \frac{\omega\,s}{\beta^2 c_0}\frac{w_0^2}{c_0^2}\sin\sigma_0\,(r\cos\varphi - s\cos\sigma_0) + \frac{\omega\,s}{\beta^2 c_0}\frac{u_0\,w_0}{c_0^2}(x - k_1\chi)\cos\sigma_0\right]^{-1};$$

where

$$\sigma_0 = \varphi_0 + \chi - \frac{\omega\,u_0}{\beta^2 c_0^2}(x - k_1\chi) - \frac{\omega\,w_0}{\beta^2 c_0^2}(r\sin\varphi - s\sin\sigma_0) + \frac{\omega\,D_0}{\beta^2 c_0}; \tag{92}$$

$$D_0 = \left[(x - k_1\chi)^2\left(1 - \frac{w_0^2}{c_0^2}\right) + \left(1 - \frac{u_0^2}{c_0^2}\right)(r^2 + s^2 - 2\,r\,s\cos(\varphi - \sigma_0)) - \right.$$
$$\left. - \frac{w_0^2}{c_0^2}(r\cos\varphi - s\cos\sigma_0)^2 + 2\frac{u_0\,w_0}{c_0^2}(x - k_1\chi)(r\sin\varphi - s\sin\sigma_0)\right]^{1/2}.$$

Just as in normal propeller acoustics, the pressure field p radiated by the pressure jump or the lift distribution at the rotating blades can be obtained by a dipole covering corresponding to this pressure jump along the blades. This is because, according to Equations (84) and (85), p also satisfies a wave equation of the type (85). This model corresponds to the linearized listing wing theory.

In the present investigation we limited ourselves to the lifting line theory (Sections 2 to 6). In this case, the aerodynamic force affecting the lifting line (per unit of length in the radial direction) is represented by a radial dipole distribution. The dipole axes are perpendicular to the relative incident flow velocity at the location of the lifting line. The unit vector in this dipole axis direction is apparently given by

$$e_n = \frac{\left(s + \frac{w_0}{\omega}\cos\varphi_0\right)e_x - \frac{u_0}{\omega}e_\varphi}{\sqrt{\left(s + \frac{w_0}{\omega}\cos\varphi_0\right)^2 + \left(\frac{u_0}{\omega}\right)^2}}.$$

In this way, from the basic solution (92) we obtain the pressure field p radiated over all N rotor blades by the lift distribution (that is, by the pressure jump) in the following form by dipole formation using $\Pi(s,\varphi_0)$ as the dipole moment[9]

$$\begin{aligned}
p(x,r,\varphi,\varphi_0) = p(x,y,z,\varphi_0) &= p_0 - \frac{1}{4\pi}\sum_{n=0}^{N-1}\int_{R_i}^{R_a}\left[\left(s + \frac{w_0}{\omega}\cos\varphi_n\right)^2 + \frac{u_0^2}{\omega^2}\right]^{-1/2}\times\\
&\times\left[\left(s + \frac{w_0}{\omega}\cos\varphi_n\right)\frac{\partial}{\partial x} - \frac{u_0}{\omega s}\frac{\partial}{\partial\varphi}\right]\Pi(s,\sigma_n)\left[D_n\left(1 - \frac{w_0\,\omega\,s}{\beta^2 c_0^2}\cos\sigma_n\right) + \right.\\
&\left. + \frac{\omega\,r\,s}{\beta^2 c_0}\left(1 - \frac{u_0^2}{c_0^2}\right)\sin(\varphi - \sigma_n) + \frac{\omega\,s}{\beta^2 c_0}\frac{w_0^2}{c_0^2}\sin\sigma_n\,(r\cos\varphi - s\cos\sigma_n) + \frac{\omega\,s}{\beta^2 c_0}\frac{u_0 w_0}{c_0^2}x\cos\sigma_n\right]^{-1} ds;
\end{aligned}$$

with

$$\begin{aligned}
\sigma_n &= \varphi_n - \frac{\omega\,u_0}{\beta^2 c_0^2}x - \frac{\omega\,w_0}{\beta^2 c_0^2}(r\sin\varphi - s\sin\sigma_n) + \frac{\omega\,D_n}{\beta^2 c_0};\\
D_n &= \left[x^2\left(1 - \frac{w_0^2}{c_0^2}\right) + \left(1 - \frac{u_0^2}{c_0^2}\right)(r^2 + s^2 - 2\,r\,s\cos(\varphi - \sigma_n)) - \right.\\
&\left. - \frac{w_0^2}{c_0^2}(r\cos\varphi - s\cos\sigma_n)^2 + 2\frac{u_0 w_0}{c_0^2}x(r\sin\varphi - s\sin\sigma_n)\right]^{1/2};\\
\beta^2 &= 1 - \frac{u_0^2}{c_0^2} - \frac{w_0^2}{c_0^2}.
\end{aligned} \tag{93}$$

In order to form the derivatives with respect to x and we obtain the following relationships by differentiating D_n and σ_n with respect to x, φ and then performing eliminations: /308

$$\begin{aligned}
\frac{\partial D_n}{\partial x} &= \left[x\left(1 - \frac{w_0^2}{c_0^2} - \frac{w_0\,\omega\,s}{c_0^2}\cos\sigma_n\right) + \frac{r\,s\,\omega\,u_0}{c_0^2}\sin(\varphi - \sigma_n) + \frac{u_0 w_0}{c_0^2}(r\sin\varphi - s\sin\sigma_n)\right]\frac{1}{Ne};\\
\frac{\partial\sigma_n}{\partial x} &= \frac{\omega}{\beta^2 c_0}\left[x\left(1 - \frac{w_0^2}{c_0^2}\right) + \frac{u_0 w_0}{c_0^2}(r\sin\varphi - s\sin\sigma_n) - \frac{u_0}{c_0}D_n\right]\frac{1}{Ne}.
\end{aligned} \tag{94}$$

$$\begin{aligned}
\frac{\partial D_n}{\partial\varphi} &= \left[x\frac{u_0 w_0}{c_0^2}r\cos\varphi + \frac{w_0^2}{c_0^2}(r\cos\varphi - s\cos\sigma_n)\,r\sin\varphi + \right.\\
&\left. + r\,s\left(1 - \frac{u_0^2}{c_0^2}\right)\sin(\varphi - \sigma_n) + \frac{r\,s\,\omega\,w_0}{c_0^2}\sin(\varphi - \sigma_n)(r\cos\varphi - s\cos\sigma_n)\right]\frac{1}{Ne};
\end{aligned} \tag{95}$$

(9) In the lifting line theory we must set $\chi = 0$ in Equation (92).

$$\frac{\partial\sigma_n}{\partial\varphi} = \frac{\omega}{\beta^2 c_0}\left[x\frac{u_0 w_0}{c_0^2} r\cos\varphi + \frac{w_0^2}{c_0^2}(r\cos\varphi - s\cos\sigma_n) r\sin\varphi + \right.$$
$$\left. + r s\left(1 - \frac{u_0^2}{c_0^2}\right)\sin(\varphi - \sigma_n) - \frac{w_0}{c_0} D_n \cdot r\cos\varphi\right]\frac{1}{Ne}; \tag{96}$$
$$Ne = D_n\left(1 - \frac{w_0\,\omega\, s}{\beta^2 c_0^2}\cos\sigma_n\right) + \frac{\omega\, r\, s}{\beta^2 c_0}\left(1 - \frac{u_0^2}{c_0^2}\right)\sin(\varphi - \sigma_n) +$$
$$+ \frac{\omega\, s}{\beta^2 c_0}\frac{w_0^2}{c_0^2}\sin\sigma_n(r\cos\varphi - s\cos\sigma_n) + \frac{\omega\, s}{\beta^2 c_0}\frac{u_0 w_0}{c_0^2} x\cos\sigma_n .$$

When evaluating the integrands in formulas (93) for the target points of the far field (in which the radiated acoustic pressure field is of interest) no difficulties are encountered, because all the integrands are continuous.

In order to obtain the velocity potential from Equation (93), we must integrate the relationship (84). First, Equation (84) takes on the following form using the transformation (86):

$$\frac{1}{\varrho_0}[p(X\cos\lambda - Z\sin\lambda, y, X\sin\lambda + Z\cos\lambda, \varphi_0) - p_0] = \omega\frac{\partial\Phi_1}{\partial\varphi_0} - U\frac{\partial\Phi_1}{\partial X},$$

which has the well-known solution

$$\Phi_1(X, y, Z, \varphi_0) = \Phi(x, y, z, \varphi_0) = -\frac{1}{U\varrho_0}\int_{-\infty}^{X}\left[p\left(X'\cos\lambda - Z\sin\lambda, y, X'\sin\lambda + \right.\right.$$
$$\left.\left. + Z\cos\lambda, \varphi_0 + \frac{\omega}{U}(X - X')\right) - p_0\right] dX'$$
$$= -\frac{1}{U\varrho_0}\int_{-\infty}^{x\cos\lambda + z\sin\lambda}\left[p\left(X'\cos\lambda + x\sin^2\lambda - z\sin\lambda\cos\lambda, y, X'\sin\lambda - \right.\right.$$
$$\left.\left. - x\sin\lambda\cos\lambda + z\cos^2\lambda, \varphi_0 + \frac{\omega}{U}(x\cos\lambda + z\sin\lambda - X')\right) - p_0\right] dX'.$$

If we set

$$x\cos\lambda + z\sin\lambda - X' = \frac{U}{\omega}\psi,$$

then we find the following clear result

$$\Phi(x, y, z, \varphi_0) = -\frac{1}{\omega\varrho_0}\int_0^{\infty}\left[p\left(x - \frac{u_0}{\omega}\psi, y, z - \frac{w_0}{\omega}\psi, \varphi_0 + \psi\right) - p_0\right] d\psi \tag{97}$$

which gives the connection between the pressure field and the velocity potential.

If we now again consider the case of incompressible flow, then we find the following from (93) and (97) if $c_0 \to \infty$: /309

$$\Phi = -\frac{1}{4\pi}\frac{1}{\omega\varrho}\sum_{n=0}^{N-1}\int_{R_i}^{R_a}\int_{0}^{\infty}\frac{\Pi(s,\varphi_n+\psi)}{\sqrt{\left(s+\frac{w_0}{\omega}\cos(\varphi_n+\psi)\right)^2+\left(\frac{u_0}{\omega}\right)^2}}\times$$
$$\times\left[\left(x-\frac{u_0}{\omega}\psi\right)^2+\left(y-s\cos(\varphi_n+\psi)\right)^2+\left(z-s\sin(\varphi_n+\psi)-\frac{w_0}{\omega}\psi\right)^2\right]^{-3/2}\times \tag{98}$$
$$\times\left[\left(s+\frac{w_0}{\omega}\cos(\varphi_n+\psi)\right)\left(x-\frac{u_0}{\omega}\psi\right)+\frac{u_0}{\omega}y\sin(\varphi_n+\psi)-\frac{u_0}{\omega}\left(z-\frac{w_0}{\omega}\psi\right)\cos(\varphi_n+\psi)\right]d\psi\, ds\,.$$

Formula (98) exactly corresponds to the well-known (see [1], page 158) velocity potential of the linearized vortex lifting line theory of the helicopter rotor for $k_0 = u_0/\omega$ und $k_* = w_0/\omega$, as well as for the wing circulation Γ. This relationship is:

$$\Pi(s,\varphi_0) = \varrho\sqrt{(\omega s + w_0\cos\varphi_0)^2 + u_0^2}\,\Gamma(s,\varphi_0)\,. \tag{99}$$

Equation (99) corresponds to the KUTTA-JOUKOWSKI theorem. As stated above, the dipole moment Π corresponds to the force per unit of length in the radial direction along the lifting line.

Since according to the solution of the boundary value integral equation in Section 5, the blade circulation Γ is assumed to be known, Equation (99) also gives the dipole moment Π required to calculate the acoustic far field.

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